KONGUNADU COLLEGE OF ENGINEERING AND TECHNOLOGY

(Autonomous)

(Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai)

(Accredited by NBA, NAAC B++ Grade, Recognized by UGC with 2(f) & 12(B), An ISO 9001:2015 Certified Institution)

Namakkal - Trichy Main Road, Tholurpatti (Po), Thottiam (Tk), Trichy (Dt) – 621 215

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

COURSE MATERIAL

24MA301- DISCRETE MATHEMATICS AND LINEAR PROGRAMMMING

Regulation: R-2024

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DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

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COURSE FACULTY **HOD**

UNIT-I

PROPOSITIONAL CALCULUS

1.0 Introduction

Logic and proofs form the foundation of Discrete Mathematics, providing a framework for reasoning and problem-solving. Logic involves the study of statements, their truth values, and logical operations such as conjunction (AND), disjunction (OR), and negation (NOT). Propositional logic and predicate logic help in structuring mathematical arguments. Proof techniques, including direct proof, proof by contradiction, and mathematical induction, validate mathematical statements. These methods are essential in computer science, cryptography, and algorithm analysis, ensuring correctness and reliability. Mastering logic and proofs enables one to construct rigorous arguments, derive conclusions, and solve complex mathematical problems systematically.

1.1 Proposition:

A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Example:

1. Delhi is the capital of India.
2. 2 +3 =5.

Remark:

Consider the following sentences.

1. What time is it?(Interrogative sentence)
2. x+y=z(Neither true nor false)
3. x+1 =2(Neither true nor false)
4. Read this carefully. (command)
5. Obey my orders.(command)
6. What a wonderful joke this is. (Exclamatory sentence)

Note:

1. If a proposition is true then its truth value is denoted by T.
2. If a proposition is false then its truth value is denoted by F.
3. P,Q,R,S…are used to denote propositions(or) p,q,r,s…are used to denote propositions.

Primary statements (Atomic statements):

Declarative sentences which cannot be further split up into simple sentences are called primary statements or atomic statements or primitive statements.

Example:

Mani is an assistant professor.

Compound statements (Molecular statements or Composite statements):

Statements which contain one or more primary statements and some connectives are called compound statements or molecular statements or composite statements.

Example:

1. Abinesh and Bala passed the examination.
2. Today is Monday or it is raining today.

Connectives:

Connective is an operation which is used to connect two or more than two statements. Simply it is called sentential connectives. It is also known as logical connectives or logical operators.

Negation($\neg$ or ~):

Let p be a proposition. The negation of p, denoted by $\neg p$, is the statement "It is not the case that p". The proposition $\neg p$ is read "not p". The truth value of the negation of p ($\neg p$), is the opposite of the truth value of p.

Example:

p : Today is Wednesday.

$\neg p$: Today is not Wednesday.

Conjunction:

Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition "p and q". The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Example:

p :It is raining.(T)

q :I am getting cold.(T)

$p \wedge q$: It is raining and I am getting cold.(T)

Disjunction:

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q". The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Example:

p :3 is a positive integer. (T)

q: $\sqrt{2}$ is an irrational number.(T)

$p \vee q$:3 is a positive integer or $\sqrt{2}$ is an irrational number.(T)

Truth Table for the Negation of a Proposition.

P	$\neg p$
T	F
F	T

Truth Table for the Conjunction of two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth Table for the Disjunction of two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional:

Let p and q be propositions. The conditional statement $p \to q$ is the proposition "if p, then q". The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise.

In the conditional statement $p \to q$, p is called the hypothesis (or antecedent orpremise) and q is called the conclusion (or consequence).

A conditional statement is also called an implication.

Example:

p : Maria learns discrete mathematics.

q : Maria will find a good job.

$p \to q$:If Maria learns discrete mathematics ,then she will find a good job.

Biconditional:

Let p and q be propositions. The biconditional statement p xq is the proposition "p if and only if q". The biconditional statement p xq is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.

Example:

P : You can take the flight.

q : You buy a ticket.

pxq : You can take the flight if and only if you buy a ticket.

Truth Table for the Conditional $p \rightarrow q$		
p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table for the Biconditional p xq		
p	q	p xq
T	T	T
T	F	F
F	T	F
F	F	T

Note:

Some other common ways to express $p \rightarrow q$	Some other common ways to express pxq
"if p, then q"	
"if p, q"	
"q if p"	
"q when p"	"p is necessary and sufficient for q"
"q unless p"	"if p then q, and conversely"
"p implies q"	"p iff q"
"p only if q"	
"q whenever p"	
"q follows from p"	

Precedence of logical operators:

1. The conjunction operator takes precedence over the disjunction operator.
2. Theconditionalandbiconditionaloperatorshavelowerprecedencethanthe conjunction and disjunction operators.
3. The conditional operator has precedence over the biconditional operator.

Table for precedence of logical operators	
Operator	Precedence
$\neg$	1
$\wedge$ v	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Translating English sentences:

Example:

How can this English sentence be translated into a logical expression?

"You can access the internet from campus only if you are a computer science major or You are not a freshman."

Solution:

Let a, c and f represent "You can access the Internet from campus ","You are a computer science major" and "You are a freshman" respectively.

Then the sentence can be translated to $a \rightarrow (c \vee \neg f)$.

Example:

Express the specification "The automated reply cannot be sent when the file system is full" using logical connectives.

Solution:

Let p and q represent "The automated reply can be sent" and "The file system is full" respectively. Then $\neg p$ represents "It is not the case that the automated reply can be sent", which can also be expressed as "The automated reply cannot be sent."

Therefore the given sentence can be translated to $q \rightarrow \neg p$.

Example:

How can this English sentence be translated into a logical expression?

"You cannot ride the roller coaster if you are under 4feet tall unless you are older than 16 years old."

Solution:

Let p, q and r represent "You can ride the roller coaster", "You are under 4 feet tall" and "You are older than 16 years old" respectively.

Then the sentence can be translated to $(q \wedge \neg r) \rightarrow \neg p$.

Example:

Write the following sentence in symbolic form, if either Kumar takes calculus or Suresh takes Sociology, then Ramesh will take English.

Converse, contrapositive and inverse:

The proposition $q \rightarrow p$ is called the converse of $p \rightarrow q$.

The contrapositive of $p \rightarrow q$ is the proposition $\sim q \rightarrow \sim p$.

The proposition $\sim p \rightarrow \sim q$ is called the inverse of $p \rightarrow q$.

Note:

Only the conditional proposition and its contrapositive are logically equivalent.

Example:

What are the contrapositive, the converse and the inverse of the conditional statement "The home team wins whenever it is raining".

Solution:

We know that "q whenever p "is one of the ways to express the conditional statementp $\rightarrow q$. Therefore the given statement can be rewritten as "If it is raining, then the home team wins".

The contrapositive of this conditional statement is "If the home team does not win, then it is not raining."

The converse is "If the home team wins, then it is raining".

The inverse is "If it is not raining, then the home team does not win".

1. Write the following statements in symbolic form: If Ravi is not in a good mood or he is not busy, then he will go to New Delhi.

Solution:

P: Ravi is in a good mood.

$\neg P$: Ravi is not in a good mood.

Q: He is busy.

$\neg Q$: He is not busy.

R: Go to New Delhi.

mSymbolic form$(\neg P \vee \neg Q) \rightarrow R$.

2. Write the logical expression for "If tigers have wings then the earth travels round the sun"

Solution:

P: Tigers have wings.

Q: Earth travels round the sun.

mThe logical expression is $P \rightarrow Q$.

Converse, contrapositive and inverse statement

If $P \rightarrow Q$ is conditional statement then

(i) $Q \rightarrow P$ is called converse of $P \rightarrow Q$

(ii) $\neg Q \rightarrow \neg P$ is called contrapositive $P \rightarrow Q$

(iii) $\neg P \rightarrow \neg Q$ is called inverse of $P \rightarrow Q$

3. Give the converse and the contrapositive of the implication "If it is raining, then I get wet".

Solution:

P: It is training

Q: I get wet.

Converse: $Q \rightarrow P$ [If I get wet, then it is raining]

Contrapositive: $\neg Q \rightarrow \neg P$[If I do not get wet, then it is not raining]

4. What are the contrapositive,the converse, and the inverse of the conditional statement? "If you work hard then you will be rewarded".

Solution:

P :You work hard / $\neg P$: you will not work hard

Q: You will be rewarded / $\neg Q$: you will not be rewarded

Converse: $Q \rightarrow P$

You will be rewarded only if you work hard

Contra positive: $\neg Q \rightarrow \neg P$

If you will not be rewarded then you will not work hard.

Inverse: $\neg P \rightarrow \neg Q$

If you will not work hard then you will not be rewarded.

Construction of Truth table

5. Write the truth table for $(P \wedge Q) \rightarrow (P \vee Q)$

Solution:

P	**Q**	**$P \wedge Q$**	**$P \vee Q$**	**$(P \wedge Q) \rightarrow (P \vee Q)$**
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

m**Note**

$\wedge \rightarrow$ **T T T**

$\vee \rightarrow$ **FFF**

$\rightarrow$ **T FF**

6. Construct the truth table for the following

a) $P \wedge (P \vee Q)$

b) $P \rightarrow \tilde{Q}$

Solution:

a) $P \wedge (P \vee Q)$

P	**Q**	**$P \vee Q$**	**$(P \wedge (P \vee Q)$**
T	T	T	T
T	F	T	T
F	F	T	F
F	F	F	F

b) $P \rightarrow \tilde{Q}$

P	**Q**	**$P \vee Q$**	**$(P \wedge (P \vee Q)$**
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

7. Construct the truth table for the compound proposition$(P\rightarrow\tilde{Q})\leftrightarrow(\neg P\rightarrow\neg Q)$

Solution:

P	**Q**	**¬P**	**¬Q**	**P→ Q**	**¬P→¬Q**	**(P→Q) ↔(¬P→¬Q)**
T	T	F	F	T	T	T
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

1.2 Propositional equivalences:

Tautology:

A compound proposition that is always true ,no matter what the truth values of the propositions that occur in it, is called a tautology.

Example:

$p \vee \neg p$ is a tautology.

Contradiction:

A compound proposition that is always false, no matter what the truth values of the propositions that occur in it, is called a contradiction.

Example:

$p \wedge \neg p$ is a contradiction.

Contingency:

A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Example:

$p\rightarrow q$ is a contingency.

Problems:

1. Show that$(p\rightarrow(q\rightarrow r))\rightarrow((p\rightarrow q)\rightarrow(p\rightarrow r))$ is a tautology.

Solution:

Let S: $(p\rightarrow(q\rightarrow r))\rightarrow((p\rightarrow q)\rightarrow(p\rightarrow r))$.

p	q	r	p→q	p→r	q→r	(p→(q→r))	((p→q)→(p→r))	S
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T

T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

From the truth table, all possible truth values, the last column is true.

Therefore the given statement formula is a tautology.

2. Show that (pq)((pvq)) is a contradiction.

Solution:

Given (pq)((pvq)).

p	q	pq	pvq	(pvq)	(pq)((pvq))
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

From the truth table, all possible truth values, the last column is false.

Therefore the given statement formula is a contradiction.

Logically equivalent:

Compound propositions that have the same truth values in all possible cases are called logically equivalent. We can also define this notion as follows.

The compound propositions p and q are called logically equivalent if pxq is a tautology. The notation p Ξq denotes that p and q are logically equivalent.

Example:

(p→q) (q→p) and (pq)v(p q) are logically equivalent.

Problems:

1. Show that P is equivalent to the following propositions

i). (P Q)v(P   Q)and

ii). (PvQ) (P v Q)

Solution:

Let us construct the truth table for the given statement formulas (PQ)v(PQ) and (P vQ) (P vQ).

P	Q	¬Q	P∧Q	P∧¬Q	(P∧Q)v(P∧¬Q)	PvQ	Pv¬Q	(PvQ)∧(Pv¬Q)
T	T	F	T	F	T	T	T	T
T	F	T	F	T	T	T	T	T
F	T	F	F	F	F	T	F	F
F	F	T	F	F	F	F	T	F

From the above truth table, Pis equivalent to (P∧Q)v(P∧¬Q) and (PvQ)∧(Pv¬Q).

2. Using truth table, show that (PxQ)e(P∧Q)v(¬P∧¬Q).

Logical Equivalences	
Equivalence	**Name**
p∧T ≡ p p vF ≡ p	Identity laws
pvT ≡T p ∧F≡F	Domination laws
pvp ≡ p p∧p ≡ p	Idempotent laws
¬(¬p)≡p	Double negation law
pvq ≡ qvp p∧q ≡ q∧p	Commutative laws
(pvq)vr ≡ pv(qvr) (p∧q)∧r≡p∧(q∧r)	Associative laws
pv(q∧r) ≡ (pvq) ∧ (pvr) p∧(qvr) ≡ (p∧q) v (p ∧r)	Distributive laws
¬(p∧q) ≡ ¬p v¬q ¬(pvq)≡¬p ∧¬q	De Morgan's laws
Pv(p∧q) ≡ p p∧(p vq)≡p	Absorption laws
pv¬p ≡T p∧¬p ≡F	Negation laws (Complement laws)
P→(q→r) ≡ (P∧q)→r	Exportation law

Logical Equivalences Involving Conditional Statements	Logical Equivalences Involving Biconditional Statements
p →q Ξ□p vq p →q Ξ□q→□p p vq Ξ□p →q p□ q Ξ □ (p → □ q) □(p →q) Ξ p□□q (p →q)□(p →r) Ξ p→(q □ r) (p →r)□(q →r) Ξ (p v q)→r (p →q)v(p →r)Ξ p→(q v r) (p →r)v(q →r) Ξ (p□ q)→r	P xq Ξ (p →q)□ (q →p) P xq Ξ □ p x □ q P xq Ξ (p □ q)v(□p□□q) □ (p xq) Ξ p x □ q

Implications	
(p□ q) □ p (p□ q) □ q	Simplification
p □ (p v q) q □ (p v q)	Addition
(p)□ (q)□ p □ q	Conjunction
p□(p →q) □q	Modus ponens
□ q □ (p →q) □ □ p	Modus tollens
(p →q)□(q →r) □p →r	Hypothetical syllogism
(p v q) □ □ p □ q	Disjunctive syllogism
(p v q)□(p →r)□(q →r) □r	Dilemma
(p v q)□(□p vr) □q vr	Resolution

Problems:

1. Showt hat□(P□Q)→ (□Pv (□PvQ))e□PvQ.

Solution:

□(P□Q)→(□P v(□P vQ))

e□(P□Q)→((□Pv□P)vQ) (Associative law)

e□ (P □ Q)→(□ P v Q) (Idempotent law)

e □ (□ (P □ Q))v(□ P v Q) (p →q Ξ□p vq)

e(P Q)v(P v Q) e(Pv(PvQ))(Qv	(Double negation law)
(PvQ))	(Double negation law)
e(P v P)vQ) (Qv(Q v P))	(Associative law and commutative law)
e(TvQ)((QvQ)vP)	(Complement law and associative law)
e T (Q v P) e P v Q	(Domination law and idempotent law)
Hence (P Q)→(P v (P v Q))e P v Q .	(Identity law and commutative law)

2. Show that(Q(P→Q))→P is a tautology.

Solution:

(Q (P→Q))→ P

Ξ (Q (P v Q))→ P	
Ξ ((Q P)v(Q Q))→ P	(p →q Ξp vq)
Ξ ((Q P)vF)→ P	(Distributive law)
Ξ (QP)→P	(Domination law)
Ξ (QP)vP Ξ(	(Identity law)
QvP)v P	(p →q Ξp vq) (De
Ξ Qv(P v P)	Morgan‟s law)
Ξ QvT	(Associative law)
Ξ T	(Complement law)-
Hence(Q(P→Q))→P is a tautology.	(Domination law)

Tautological implication:

A statement formula A is said to be tautologically imply a statement formula B if and only if A →B is tautology. We shall denote this idea by A B which is read as “A implies B”.

Note:

1. A B if and only if A→B is tautology.(i.e)To prove AB ,it is enough to prove A→B is tautology.
2. → means connective conditional.

x means connective biconditional.

e means equivalent.

 means tautological implication.

Problems:

1. **Prove that** (p→(q →r)) ((p →q)→(p →r)) **by constructing truth table.**

Solution:

To prove that (p→(q → r)) ((p → q)→(p → r)).

It is enough to prove that S:(p→(q → r))→((p → q)→(p →r)) is a tautology.

P	q	R	q → r	p→ q	p→ r	(p→(q→ r))	(p→ q)→(p→ r)	S
T	T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	F	T
T	F	T	T	F	T	T	T	T
T	F	F	T	F	F	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Hence (p →(q →r)) →((p →q) →(p →r))is a tautology.

Therefore (p→(q→ r)) ((p→ q)→(p→ r)).

2. Prove that ((pvq)(p →r)(q →r))r.

Solution:

To prove that((p v q) (p → r) (q → r)) r.

It is enough to prove that ((p v q) (p →r) (q →r))→r is a tautology.

((p v q)(p →r)(q →r))→r

Ξ ((p v q)(p v r)(q v r))→r (p →q Ξp vq)

Ξ ((p v q)((p q)vr))→ r (Distributive law)

Ξ ((p v q) ((p v q)vr))→ r (De Morgan's law)

Ξ (((p v q)(p v q))v((p v q) r))→r (Distributive law)

Ξ (Fv((p v q) r))→r	(Complement law)
Ξ ((p v q) r)→r	(Identity law)
Ξ ((p v q)r)vr	(p →q Ξp vq) (De
Ξ ((p v q)v r)vr Ξ	Morgan‟s law)
(p vq)v(r v r)	(Associative law)
Ξ (p v q)vT	(Complement law)
ΞT	(Domination law)

Hence((p v q) (p →r) (q →r))→ r is a tautology.

Therefore ((p v q) (p →r) (q →r)) r .

3. Using truth table, show that the proposition P∨˥(P∧Q) is a tautology.

Solution:

p	***Q***	**P∧Q**	**˥(P∧Q)**	**P∨˥(P∧Q)**
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Since all the entries in the resulting column is T.

∴The given proposition is a tautology.

4. Is(˥P∧(P∨Q))→Q a tautology?

Solution:

p	***Q***	˥P	**P∨Q**	˥P**∧**(P∨Q)	**(˥P∧(P∨Q))→Q**
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Since all the entries in the resulting column is T.

∴The given proposition is a tautology.

5. Prove that (˥P∧P)∧Q is a contradiction.

Solution:

p	***Q***	˥P	**˥P∧P**	**(˥P∧P)∧Q**
T	T	F	F	F
T	F	F	F	F
F	T	T	F	F
F	F	T	F	F

Since all the entries in the resulting column is F.

∴The given proposition is a contradiction.

6. Prove that (˥P→Q)→ (Q→P) is a contradiction or tautology.

Solution:

P	***Q***	˥P	**˥P→ Q**	Q→P	**(˥P→Q)→(Q→P)**
T	T	F	T	T	T
T	F	F	T	T	T
F	T	T	T	F	F
F	F	T	F	T	T

Since all the entries in the resulting column is neither T nor F.

∴The given proposition is an either tautology nor contradiction.

7. Using truth tables how that P∨(P∧Q)≡P(or) show that P is equivalent to P∨(P∧Q)

Solution:

p	***Q***	**P∧P**	**P∨(P∧Q)**
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

∴From column1&4 P≡P∨(P∧Q).

8. Show that the following propositions are logically Equivalent $P\to Q$ & $\neg P \vee Q$.

Solution

p	*Q*	$\neg P$	$\neg P \vee Q$	**P→ Q**
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

From column 4&5 we get $P\to Q$ and $\neg P\vee Q$ are equivalent.

ie: $P \to Q \equiv \neg P\vee Q$ (or) $P \to Q \rightleftharpoons \neg P\vee Q$.

9. Show that $\sim(P\vee Q) \rightleftharpoons \sim P\wedge \sim Q$

Solution:

P	*Q*	~P	~Q	~P**Λ**~Q	**P∨Q**	**~(P∨Q)**
T	T	F	F	F	T	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	T	F	T

From column 5 &7 weget $\sim(P\vee Q)\rightleftharpoons\sim P\wedge \sim Q$.

10. Show that $(\neg P \wedge (\neg Q \wedge R))\vee(Q \wedge R)\vee (P \wedge R)\rightleftharpoons R$

Solution:

(¬P Λ (¬Q Λ R))∨(Q Λ R)∨(P Λ R)	***Reasons***
$\rightleftharpoons(\neg P \wedge (\neg Q \wedge R))\vee((Q\vee P) \wedge R)$	∴Distributive laws
$\rightleftharpoons((\neg P \wedge \neg Q) \wedge R)\vee((Q\vee P) \wedge R)$	Associative laws
$\rightleftharpoons((\neg P \wedge \neg Q)\vee(Q\vee P)) \wedge R$	Distributive laws
$\rightleftharpoons(\neg (P \vee Q)\vee(Q\vee P))A \wedge R$	Demorgan's laws
$\rightleftharpoons(\neg (P \vee Q)\vee(P\vee Q)) \wedge R$	Commutative laws
$\rightleftharpoons T \wedge R$	$P \wedge \neg P \rightleftharpoons T$
R	$P \wedge T\rightleftharpoons P$

∴ $(\neg P \wedge (\neg Q \wedge R))\vee(Q \wedge R) \vee(P \wedge R) \rightleftharpoons R$. Hence the proof.

1.3 Other connectives:

NAND:

The word “NAND” is a combination of “NOT” and “AND”. The connective “NAND” Is denoted by the symbol $\uparrow$.For any two formula P and Q, $P \uparrow Q$ is defined as

$P \uparrow Q \Leftrightarrow \neg(P \wedge Q)$.

NOR:

The word “NOR” is a combination of “NOT” and “OR” .The connective “NOR” is Denoted by the symbol $\downarrow$.For any two formula P and Q, $P \downarrow Q$ is defined as

$P \downarrow Q \Leftrightarrow \neg(P \vee Q)$.

TruthTablefor “NAND”		
p	q	$\neg(p \wedge q)$
T	T	F
T	F	T
F	T	T
F	F	T

TruthTablefor “NOR”		
p	q	$\neg(p \vee q)$
T	T	F
T	F	F
F	T	F
F	F	T

Dual of a statement formula:

Two formulas A and A^* are said to be duals of each other, if either one can be Obtained from other by replacing $\wedge$ by v,v by $\wedge$,T by F and F by T.

Example:

Dual of $(\neg p \wedge q) \vee T$ is $(\neg p \vee q) \wedge F$.

Duality principle theorem:

If A and A^* are the dual statements and if $P_1, P_2, \ldots, P_n$ are the atomic variables that Occur in A and A^*.i.e., $A = A(P_1, P_2, \ldots, P_n)$

$A^* = A^*(P_1, P_2, \ldots, P_n)$

i). $\neg A(P_1, P_2, \ldots, P_n) \Leftrightarrow A^*(\neg P_1, \neg P_2, \ldots, \neg P_n)$.

ii). $A(\neg P_1, \neg P_2, \ldots, \neg P_n) \Leftrightarrow \neg A^*(P_1, P_2, \ldots, P_n)$.

That is, negation of a statement is equivalent to its dual in which every variable is replaced by its negation.

Note:

If any two statements A and B are equivalent ,then duals $A^*$and $B^*$are also equivalent.

Normal forms:

If we write the given statement formula in a particular form (in terms of $\wedge$, v and $\neg$), then it is called normal form.

Here given statement formula and its normal form are equivalent.

Note:

It will be convenient to use the word "product" in place of "conjunction" and "sum" in Place of "disjunction".

Elementary product:

A product of the statement variables and their negations in a formula is called elementary product.

Example:

Let P and Q be any two atomic variables. Then some possible elementary products are P,Q,$\neg P$,$P \wedge Q$,$P \wedge \neg Q$, $P \wedge \neg P$.

Elementary sum:

A sum of the statement variables and their negations in a formula is called elementary sum.

Example:

Let P and Q be any two atomic variables. Then some possible elementary sums are P, Q, $\neg P$, P v$\neg$Q, $\neg$P v$\neg$Q.

Remark:

A statement variable alone is an elementary product (sum), because P can be written as $P \wedge P$ (P vP).

Disjunctive normal form(DNF):

For a given statement formula, an equivalent formula consisting of sum of elementary products only is known as its disjunctive normal form.

Example:

$(P \wedge Q)$ v$(P \wedge Q \wedge \neg P)$ v $(P \wedge \neg Q)$ is a disjunctive normal form.

Conjunctive normal form(CNF):

For a given statement formula, an equivalent formula consisting of product of elementary sums only is known as its conjunctive normal form.

Example:

(PvQ) □(PvQ v□P) □(Pv□Q)□P is a conjunctive normal form.

Problems:

1. Obtain a DNF of p→((p →q)□□(□qv□p)).

Solution:

Let S:p→((p →q)□ □ (□ q v □ p)).

Se p→((p →q)□□(□(q □ p)))(De Morgan‟s law)

ep→((□ p v q)□ (q □ p)) (p →q e □ p v q and Double negation law)

ep→((□ p □ q □ p)v(q □ q □ p)) (Distributive law)

e □ p v (□ p □ q □ p) v (q □ p) , (p →q e □pv q and idempotent law)

which is the required DNF.

2. Obtain a CNF of Qv(P□□Q)v(□P□□Q).

Solution:

Let S:Qv (P □□Q)v (□P □□Q).

SeQv((P v □ P)□ □ Q) (Distributive law)

e (Q vP v □ P) □ (Q v □ Q) , (Distributive law)

which is the required CNF.

Minterms:

Let P and Q be two statement variables. Let us construct all possible formulas which consist of conjunction of P or its negation and Q or its negation. None of the formulas should contain both a variable and its negation. Using commutative law if any two terms are equivalent, choose any one of the term. Collect the remaining terms. They are called minterms.

Example:

Let P and Q be two variables. Then the minterms are P □ Q ,P □ □ Q ,□ P □ Q ,□ P □ □ Q .

Maxterms:

Let P and Q be two statement variables. Let us construct all possible formulas which consist of disjunction of P or its negation and Q or its negation. None of the formulas should contain both a variable and its negation. Using commutative law if any two terms are equivalent, choose any one of the term. Collect the remaining terms. They are called maxterms.

Example:

Let P and Q be two variables. Then the maxterms are $P \vee Q$, $P \vee \neg Q$, $\neg P \vee Q$, $\neg P \vee \neg Q$.

Note:

1. If there are 'n' variables, then the number of minterms (maxterms) is 2^n.
2. In elementary product (sum) a variable and its negation exist. But in minterm (maxterm) such things does not exist.
3. Let P,Q and R be three variables. Then possible minterms and maxterms are the following table.

Minterms	**Maxterms**
$P \wedge Q \wedge R$	$P \vee Q \vee R$
$P \wedge Q \wedge \neg R$	$P \vee Q \vee \neg R$
$P \wedge \neg Q \wedge R$	$P \vee \neg Q \vee R$
$\neg P \wedge Q \wedge R$	$\neg P \vee Q \vee R$
$\neg P \wedge \neg Q \wedge R$	$\neg P \vee \neg Q \vee R$
$\neg P \wedge Q \wedge \neg R$	$\neg P \vee Q \vee \neg R$
$P \wedge \neg Q \wedge \neg R$	$P \vee \neg Q \vee \neg R$
$\neg P \wedge \neg Q \wedge \neg R$	$\neg P \vee \neg Q \vee \neg R$

Principal disjunctive normal form(PDNF):

For a given statement formula, an equivalent formula consisting of disjunction of minterms only is known as its principal disjunctive normal form.

Example:

$(P \wedge Q) \vee (\neg P \wedge Q) \vee (P \wedge \neg Q)$ is a principal disjunctive normal form.

Principal conjunctive normal form(PCNF):

For a given statement formula, an equivalent formula consisting of conjunction of maxterms only is known as its principal conjunctive normal form.

Example:

(PvQvR)□(□PvQvR)□(Pv□Qv□R) is a principal conjunctive normal form.

Working rule to obtain PDNF:

Step1:By applying various equivalence rules simplify the given statement formula, if possible. i.e., Rewrite the given statement formula in terms of v, □ and □ alone.

Step2:Apply the each term □T, instead of T, apply Pv□P.

Step 3: Apply distributive laws.

Step 4: Apply commutative laws.

Working rule to obtain PCNF:

Step1:By applying various equivalence rules simplify the given statement formula,ifpossible. i.e., Rewrite the given statement formula in terms ofv, □ and □ alone.

Step2: Apply the each term vF, instead of F, apply P□□P.

Step 3: Apply distributive laws.

Step 4: Apply commutative laws.

Remark:

If the principal disjunctive (conjunctive) normal form of a given statement formula S Containing 'n' variables is known, then the principal disjunctive(conjunctive)normal form of □S will consist the disjunction (conjunction) of the remaining minterms (maxterms) which do not appear in the principal disjunctive (conjunctive) normal form of S. From Se□(□S) one can obtain the principal conjunctive (disjunctive) normal form of S.

Working rule to find PDNF(PCNF) using truth table:

Step1: Construct truth table for the given statement formula.

Step2: Choose each and every row in which the final column value is'T'('F').

Step 3: In the selected row, if the truth value of each individual variable value is 'T' ('F') select that variable and truth value is 'F' ('T'), select the negation of the variable. In such a way collect all possible minterms (maxterms).

Step4: Sum (product) of all minterms (maxterms) gives the required PDNF(PCNF).

Problems:

1. Find the principal disjunctive normal form of the statement (qv(p□r))□□((pvr)□q).

Solution:

LetS:(qv(p□r))□□((pvr)□q).

Se(qv(p□r))□(□(pvr)v□q) (DeMorgan‟s law)

e((qv(p□r))□(□p□□r))v((qv(p□r))□□q) (Distributive law)

e(q□□p□□r)v(p□r□□p□□r)v(q□□q)v(p□r□□q) (Distributive law)

e(□p□q□□r)v(p□□p□r□□r)v(q□□q)v(p□□q□r) (Commutative law)

e(□p□q□□r)vFvFv(p□□q□r) e(□p□q□□r)v(p□□q□r) (Complement law)

This is the required PDNF. (Identity law)

2. Obtain the PCNF of S:(□P→R)□(QxP).Hence obtain PDNF.

Solution:

Let S:(□P→R)□(QxP).

Se(□(□P)vR)□((Q→P)□(P→Q))

(p→qΞ□pvq and pxqΞ(p→q)□(q→p)

e(PvR)□(□QvP)□(□PvQ)

(Double negation law and p →qΞ □pvq)

e(PvRvF)□(□QvPvF)□(□PvQvF) (InsertvF)

e(PvRv(Q□□Q))□(□QvPv(R□□R))□(□PvQv(R□□R))(P□□PeF)

e(PvRvQ)□(PvRv□Q)□(□QvPvR)□(□QvPv□R)□(□PvQvR)

□(□PvQv□R) (Distributive law)

e(PvQvR)□(Pv□QvR)□(Pv□Qv□R)□(□PvQvR)□(□PvQv□R)
(Commutative law)

This is the required PCNF.

To find PDNF:

The remaining maxterms are PvQv□R,□Pv□QvR and □Pv□Qv□R.

Therefore □Se(PvQv□R)□(□Pv□QvR) □(□Pv□Qv□R)

Taking negation on both sides, we have

□(□S)e□((PvQv□R)□(□Pv□QvR)□(□Pv□Qv□R))

Se□(PvQv□R)v□(□Pv□QvR)v□(□Pv□Qv□R)

(□(□P)eP and DeMorgan's law)

Se(□P□□Q□R)v(P□Q□□R)v(P□Q□R),which is the required PDNF.

3. Obtain the PDNF and PCNF of (P□Q)v(□P□R)v(Q□R),using truth table.

Solution:

Let S:(P□Q)v(□P□R)v(Q□R).

P	***Q***	***R***	□P	P□Q	□P□R	Q □R	***S***	**Minterms**	**Maxterms**
T	T	T	F	T	F	T	T	P□Q□R	-
T	T	F	F	T	F	F	T	P□Q□□R	-
T	F	T	F	F	F	F	F	-	□PvQv□R□PvQvR
T	F	F	F	F	F	F	F	-	-
F	T	T	T	F	T	T	T	□P□Q□R	Pv□QvR
F	T	F	T	F	F	F	F	-	-
F	F	T	T	F	T	F	T	□P□□Q□R	PvQvR
F	F	F	T	F	F	F	F	-	

Hence the required PDNF is (P□Q□R)v(P□Q□□R)v(□P□Q□R)v (□P□□Q□R) and PCNF is (□PvQv□R)□(□PvQvR)□(Pv□QvR)□(PvQvR).

Direct method of proof:

When a conclusion is derived from a set of premises by using the accepted rules of reasoning then such a process of derivation is called a direct proof.

Indirect method of proof:

1. Method of contradiction and
2. Method of contrapositive.

Method of contradiction:

In order to show that a conclusion C follows logically from the premises $H_1,H_2,...,H_m$,we assume that C is false and consider □C as an additional premises. If the new set of premises gives contradict value, then the assumption □C is true does not hold simultaneously with H_1□H_2□...□H_m being true. Therefore C is true whenever H_1□H_2□...□H_m is true. Thus C follows logically from the premises $H_1,H_2,...,H_m$.

Method of contrapositive:

In order to prove H$_1$H$_2$...H$_m$C, if we prove C(H$_1$H$_2$...H$_m$), then the original problem follows. This method is called contrapositive method.

The theory of inferences:

The analys is of the validity of the formula from the given set of premises by using derivation is called “theory of inferences”.

4. Obtain DNF and CNF for ˥(PVQ)⇌(PΛQ)

Solution:

To prove DNF(‘Λ’ inside, ‘V’ outside)

˥***(PVQ)⇌(PΛQ)***	***Reasons***
(PVQ)⇌(PΛQ)	R↔S
e[˥(PVQ)Λ(PΛQ)]V[(PVQ)Λ˥(PΛQ)]	e(RΛS)V (˥RΛ˥S)
e[(˥P Λ ˥Q)Λ (PΛQ)]V[(PVQ)Λ(˥PV˥Q)]	Demorgan’sl aw
e[˥PΛ ˥QΛ PΛQV(PVQ) Λ(˥P V ˥Q)]	Associative law
e(˥PΛ˥QΛ PΛQ)V(P Λ(˥P V˥Q)) V(QΛ(˥P V˥Q))	Distributive rule
e(PΛQΛ˥PΛ˥Q)V(PΛ˥P)V(PΛ˥Q)V(QΛ˥P)V(QΛ˥Q)	Distributive rule
Which is required DNF	

To find CNF (‘V’ inside, ‘Λ’)

˥(PVQ)⇌(PΛQ)	***Reasons***
˥(PVQ)⇌(PΛQ)	R↔S
e[˥(PVQ)→(PΛQ)]Λ[(PΛQ)→˥(PVQ)]	e(R→S)Λ(S→R)
e(PVQ) V(PΛQ)Λ˥(PΛQ)V ˥(PVQ)	P→Q˥P VQ
e(PVQ) V(PΛQ)Λ(˥P V˥Q)V˥(PVQ)	Demorgan’s law
e(PVQVP)Λ(PVQVQ)Λ (˥PV˥Q)V(˥PΛ˥Q)	Distributive and
e(PVQVP) Λ(PVQVQ)Λ(˥P V˥QV˥P)Λ(˥PV˥QV ˥Q)	Demorgan’s law
e(PVPVQ)Λ(PVQVQ)Λ(˥P V˥PV ˥Q)Λ(˥P V ˥QV˥Q)	Distributive rule
e(PVQ) Λ(PVQ) Λ(˥P V˥P) Λ(˥P V ˥Q)	Commutative law
e(PVQ)Λ((˥PV˥P)	
Which is the required CNF	Idempotent law PΛPeP

5. Obtain Principal disjunctive normal forms of (P∧Q)∨(¬P∧ R)∨(Q∧ R)

Solution

(P∧Q)∨(¬P∧R)∨(Q∧R)	**Reasons**
⇔((P∧Q)∧T)∨(¬P ∧R)∨[(Q∧R)∧T]	P∧T ⇔P
⇔((P∧Q)∧(R∨¬R)∨[(¬P∧R)∧(Q∨¬Q)]∨ [(Q∧R)∧ (P∨¬P)]	P∨¬P⇔T
⇔(P∧Q∧R)∨(P∧Q∧¬R)∨(¬P∧R∧Q)∨ (¬P∧R∧¬Q)∨(Q∧R∧P)∨(Q∧R∧¬P)	Distributive rule
⇔(P∧Q∧R)∨(P∧Q∧¬R)∨(¬P∧¬Q∧R)∨ (P∧Q∧R) ∨ (¬P∧Q∧R)	Commutative law
⇔(P∧Q∧R)∨(P∧Q∧¬R)∨(¬P∧Q∧R)∨ (¬P∧¬Q∧R)	Idempotent rule

6. Obtain the PDNF of (P∧Q)∨(¬P∧R)∨(Q∧R) using truth table

Solution:

P	***Q***	***R***	**¬P**	**P∧Q**	**¬P∧R**	**Q∧R**	**(P∧Q)∨(¬P∧R)∨ (Q∧R)**	**Minterm**
T	T	T	F	T	F	T	T	P∧Q∧R
T	T	F	F	T	F	F	F	P∧Q∧R
T	F	T	F	F	F	F	F	-
T	F	F	F	F	F	F	F	-
F	T	T	T	F	T	T	T	¬P∧Q∧R
F	T	F	T	F	F	F	F	-
F	F	T	T	F	T	F	T	¬P∧¬Q∧R
F	F	F	T	F	F	F	F	-

∵The PDNF is(P∧Q∧R)∨(P∧Q∧¬R)∨(¬P∧Q∧R)∨(¬P∧¬Q∧R)

1.4 Rules for inferences theory:

Rule P:

A given premise may be introduced at any stage in the derivation.

Rule T:

A formula S may be introduced in a derivation if S is tautologically implied by one or more of the preceding formulae in the derivation.

Rule CP (Rule of conditional proof or deduction theorem):

If we can derive S from R and a set of given premises, then we can drive R $\rightarrow$ S from the set of premises alone. In such a case R is taken as an additional premise (assumed premise). Rule CP is also called the deduction theorem.

Note:

In general, whenever conclusion is of the form R $\rightarrow$ S (interms of conditional), we should apply rule CP. In such case, R is taken as an additional premise and S can be derived from the given premises and R.

Indirect method of derivation:

Whenever the assumed premise is used in the derivation, then the method of derivation is called indirect method of derivation.

Implications		
I_1 I_2	(p□q)□p (p □ q) □ q	Simplification
I_3 I_4	p□(pvq) q□(pvq)	Addition
I_5	□p□(p→q)	
I_6	q□(p→q)	
I_7	□(p→q)□p	
I_8	□(p→q)□□q	
I_9	(P)□(q)□p□q	Conjunction
I_{10}	□p□(pvq)□q	Disjunctive syllogism
I_{11}	p□(p→q)□q	Modus ponens

I_{12}	$\neg q \wedge (p\to q) \Rightarrow \neg p$	Modus tollens
I_{13}	$(p\to q)\wedge(q\to r) \Rightarrow p\to r$	Hypothetical syllogism
I_{14}	$(pvq)\wedge(p\to r)\wedge(q\to r)\Rightarrow r$	Dilemma
I_{15}	$(pvq)\wedge(\neg p v r)\Rightarrow q\ vr$	Resolution

Problems:

1. Show that SvR is tautologically implied by $(PvQ)\wedge(P\to R)\wedge(Q\to S)$.

Solution:

Given set of premisesare PvQ, $P\to R$ and $Q\to S$.

Conclusion is S vR .

{1}	(1)	PvQ	Rule P
{1}	(2)	QvP	Rule T, Commutative law,from(1)
{1}	(3)	$\neg Q\to P$	Rule T, $P\to Q \Leftrightarrow \neg PvQ$, from (2)
{2}	(4)	$P\to R$	Rule P
{1, 2}	(5)	$\neg Q\to R$ Q	Rule T, Hypothetical syllogism, from(3)and(4)
{3}	(6)	$\to S$	Rule P
{3}	(7)	$\neg S\to\neg Q$	Rule T, Contrapositive law, from(6)
{1, 2, 3}	(8)	$\neg S\to R$	Rule T, Hypothetical syllogism, from(5)and(7)
{1, 2, 3}	(9)		Rule T, $P\to Q \Leftrightarrow \neg PvQ$ and Double negation law, from(8)

Therefore SvR follows logically from the premises PvQ, $P\to R$ and $Q\to S$.

Thus SvR is tautologically implied by $(PvQ)\wedge(P\to R)\wedge(Q\to S)$.

2. Prove that $A\to\neg D$ is a conclusion from the premises $A\to BvC, B\to\neg A$ and $D\to\neg C$
By using conditional proof.

Solution:

Given set of premises are $A\to BvC$, $B\to\neg A$ and $D\to\neg C$.

Conclusion is $A\to\neg D$.

Consider A as an additional premise.

{1}	(1)	A	Rule P, (Assumed premise)
{2}	(2)	A→BvC	Rule P
{1, 2}	(3)	BvC	Rule T, Modus ponens, from(1)and(2)
{1, 2}	(4)	¬B→C	Rule T, P→Q⇔¬PvQ, from (3)
{1, 2}	(5)	¬C→B	Rule T, Contrapositive law, from(4) Rule P
{3}	(6)	B→¬A	Rule T, Hypothetical syllogism, from(5)and(6) Rule P
{1, 2, 3}	(7)	¬C→¬A	
{4}	(8)	D→¬C	Rule T, Hypothetical syllogism, from(7)and(8)
{1, 2, 3, 4}	(9)	D→¬A	Rule T, Contrapositive law, from (9) Rule T, Modus ponens, from (1) and (10)
{1, 2, 3, 4}	(10)	A→¬D	Rule CP
{1, 2, 3, 4}	(11)	¬D	
{1, 2, 3, 4}	(12)	A→¬D	

Therefore ¬D Follows logically from the premises A→BvC, B→¬A, D→¬C and an Additional premise A.

Thus A→¬D follows logically from the set of premises A→BvC, B→¬A and D→¬C.

3. Using indirect method show that P→Q,Q→R,PvR⇒R.

Solution:

Given set of premises are P→Q, Q→R and PvR.

Conclusion is R.

Using indirect method of proof, let us take ¬R as an additional premise.

{1}	(1)	¬R	Rule P, (Assumed premise)
{2}	(2)	P→Q	Rule P
{3}	(3)	Q→R	Rule P
{2, 3}	(4)	P→R	Rule T, Hypothetical syllogism, from(2)and(3)
{1, 2, 3}	(5)	¬P	Rule T, Modus tollens, from(1)and(4)
{4}	(6)	PvR	Rule P
{1, 2, 3, 4}	(7)	R	Rule T, Disjunctive syllogism, from(5)and(6)
{1, 2, 3, 4}	(8)	¬R∧R	Rule T, p, q⇒ p∧q, from(1)and(7)

This is nothing, but false value.

Hence P→Q,Q→R,PvR⇒R.

4. Show that the hypothesis, “It is not sunny this afternoon and it is colder than yesterday”, “we will go swimming only if it is sunny”, “If we do not go swimming, then we will takea canoe trip” and “If we take a canoe trip, then we will be home by sunset” lead to the conclusion “We will be home by sunset”.

Solution:

Let A, B, C, D and E be “It is sunny this afternoon”, “It is colder than yesterday”, “we will go swimming”, “we will take a canoe trip” and “We will be home by sunset”, respectively.

Then the given sentences can be written into symbolic form 3as, $\neg A \wedge B, C \rightarrow A$, $\neg C \rightarrow D$, $D \rightarrow E$ lead to the conclusion E.

That is ,it is enough to prove that E follows logically from the premises $\neg A \wedge B, C \rightarrow A$, $\neg C \rightarrow D$, $D \rightarrow E$.

{1}	(1)	$\neg A \wedge B$	Rule P
{1}	(2)	$\neg A$	Rule T, Simplification, from(1)
{1}	(3)	B	Rule T, Simplification, from (1)
{2}	(4)	$C \rightarrow A$	Rule P
{1, 2}	(5)	$\neg C$	Rule T, Modus tollens, from(2)and(4)
{3}	(6)	$\neg C \rightarrow D$	Rule P
{1, 2, 3}	(7)	D	Rule T, Modus ponens, from(5)and(6)
{4}	(8)	$D \rightarrow E$	Rule P
{1, 2, 3, 4}	(9)	E	Rule T, Modus ponens, from(7)and(8)

Therefore E follows logically from the premises $\neg A \wedge B$, $C \rightarrow A$, $\neg C \rightarrow D$, $D \rightarrow E$

1) Prove that the following premises are inconsistent

   (a) $P \rightarrow Q, Q \rightarrow R$, $S \rightarrow \sim R and P \wedge S$

   (b) $P \rightarrow Q, P \rightarrow R$, $Q \rightarrow \sim R and P$

2) "If I eat spicy foods, then I have strange dreams”, "I have strange dreams, if there is thunder while I sleep". " I did not have strange dreams". From these premises find the valid conclusion.

[Ans: The premises are $p \rightarrow q, q \rightarrow r, \sim q$]

Conclusion is $\sim p \wedge \sim r$.

1.5 Consistency and inconsistency of premises : Consistent:

A set of formulae $H_1, H_2, \ldots, H_m$ is said to be consistent, if their conjunction implies Tautology .i.e., $H_1 \wedge H_2 \wedge \ldots \wedge H_m \Rightarrow R \vee \neg R$, for some formula R.

Inconsistent:

A set of formulae $H_1, H_2, \ldots, H_m$ is said to be inconsistent, if their conjunction implies contradiction. i.e., $H_1 \wedge H_2 \wedge \ldots \wedge H_m \Rightarrow R \wedge \neg R$, for some formula R.

Examples:

1. Prove that the premises $a \to (b \to c), d \to (b \wedge \neg c)$ and $(a \wedge d)$ are inconsistent.

Solution:

Given premises are $a \to (b \to c), d \to (b \wedge \neg c)$ and $(a \wedge d)$.

{1}	(1)	$a \wedge d$	Rule P
{1}	(2)	A	Rule T, Simplification, from(1)
{1}	(3)	D	Rule T, Simplification, from(1)
{2}	(4)	$a \to (b \to c)$	Rule P
{1, 2}	(5)	$b \to c$	Rule T, Modus ponens, from(2)and(4)
{3}	(6)	$d \to (b \wedge \neg c)$	Rule P
{1, 3}	(7)	$b \wedge \neg c$	Rule T, Modus ponens, from(3)and(6)
{1, 3}	(8)	$\neg(\neg b \vee c)$	Rule T, De Morgan's law, from (7)
{1, 2}	(9)	$\neg b \vee c$	Rule T, $p \to q \Leftrightarrow \neg p \vee q$, from (5)
{1, 2, 3}	(10)	$\neg(\neg b \vee c) \wedge (\neg b \vee c)$	Rule T, Conjunction, from(8)and(9)

This is nothing, but false value.

Therefore the set of premises $a \to (b \to c), d \to (b \wedge \neg c)$ and $(a \wedge d)$ are inconsistent.

2. Show that the following premises are inconsistent

(i) If Tharun gets his degree, he will go for a job.

(ii) If he goes for a job, he will get married soon.

(iii) If he goes for higher study ,he will not get married.

(iv) Tharun gets his degree and goes for higher study.

Solution:

Let A,B,C and D be" Tharun gets his degree", "He will go for a job", "He will get married soon" and "He goes for higher study", respectively.

Then the given sentences can be written into symbolic form as, $A \to B, B \to C$, $D \to \neg C$, $A \wedge D$. It is enough to prove that $A \to B, B \to C, D \to \neg C, A \wedge D$ are inconsistent.

{1}	(1)	$A \to B$	Rule P
{2}	(2)	$B \to C$	Rule P
{1, 2}	(3)	$A \to C$	Rule T, Hypothetical syllogism, from(1)and(2)
{3}	(4)	$D \to \neg C$	Rule P
{3}	(5)	$C \to \neg D$	Rule T, Contrapositive law, from(4)
{1, 2, 3}	(6)	$A \to \neg D$	Rule T, Hypothetical syllogism,

{1, 2, 3} {1, 2, 3} {4} {1, 2, 3, 4}	(7) (8) (9) (10)	¬Av¬D ¬(A∧D) A∧D ¬(A∧D)∧(A∧D)	from(3)and(5) Rule T , p→q e¬pvq from (6) Rule T, De Morgan's law, from(7) Rule P Rule T, p,q ⇒p∧q, from(8)and(9)

This is nothing, but false value.

Therefore the premises A→B,B→C,D→¬C, A∧D are inconsistent.

Thus the given statements are inconsistent.

3. Demonstrate that R is a valid inference from the premises P→Q,Q→R and P.

Solution:

Here given premises are

1) P→Q

2) Q→R

3) P

Conclusion is R

	Step	Premises	Rule	Reason
	1	P→Q	Rule P	Given Premises
	2	Q→R	Rule P	Given Premises
{1,2}	3	P→ R	Rule T	P→Q,Q→R⇔P→R
	4	P	Rule P	Given Premises
{3,4}	5	R	Rule T	P,P→Q⇔Q

4. Show that R^(P∨Q) is a valid conclusion from the premises P∨Q, Q→R, P→M and ˥M

Solution:

Given premises are P∨Q, Q→R, P→M and ˥M

Conclusion is R^(P∨Q)

	Step	Premises	Rule	Reason
	1	P→M	Rule P	Given Premises
	2	˥M	Rule P	Given Premises
{1,2}	3	˥P	Rule T	˥Q,P→Q⇔˥P
	4	P∨Q	Rule P	Given Premises
{4}	5	˥P→Q	Rule T	P→Q⇔˥P∨Q

{3,5}	6	Q	Rule T	P,P→Q⇔Q
	7	Q→R	Rule P	Given Premises
{6,7}	8	R	Rule T	P, P→Q⇔Q
{4,8}	9	R^(P∨Q)	Rule T	P,Q⇔P^

5. Show that R∨S follows logically from the premises C∨D, (C∨D)→ ˥H, ˥H→ (A^˥B) and (A^˥B)→ (R∨S).

Solution:

Given premises are C∨D,(C∨D)→ ˥H,˥H→ (A^˥B)and(A^˥B)→ (R∨S).

Conclusion is R∨S

	Step	Premises	Rule	Reason
	1	C∨D	Rule P	Given Premises
	2	(C∨D)→˥H	Rule P	Given Premises
{1,2}	3	˥H	Rule T	P,P→Q⇔Q
	4	˥H→(A^˥B)	Rule P	Given Premises
{3,4}	5	A^˥B	Rule T	P,P→Q⇔Q
	6	(A^˥B)→(R∨S)	Rule P	Given Premises
{5,6}	7	R∨S	Rule T	P,P→Q⇔Q

6. Show that R→S can be derived from the premises P→(Q→S),˥R∨P and Q.

Solution:

Given premises are P→(Q→S),˥R∨P and Q.

	Step	Premises	Rule	Reason
	1	R	Assumed	Premises
	2	˥R∨P	P	Given Premises
{2}	3	R→P	T	P→Q⇔˥P∨Q
{1,3}	4	P	T	P,P→Q⇔Q
	5	P →(Q→S)	P	Given Premises
{4,5}	6	Q→S	T	P,P→Q⇔Q
	7	Q	P	Given Premises

1.6 Predicate calculus:

The predicate calculus deals with the study of predicates.

Examples:

1. Symbolize the statement “Ram is a boy”.

Solution:

Predicate is “is a boy” and subject is “Ram”.

Predicate is denoted by ‘B’.

Subject is denoted by ‘r’.

Therefore the given statement can be written into symbolic form as B(r).

n–place predicate:

In general, an n–place predicate is a predicate requiring ‘n’ names of objects, where n>0. It is denoted by $P(a_1,a_2,a_3,...,a_n)$, where $a_1,a_2,a_3,...,a_n$ are the names of the objects associated with predicate and P is a predicate.

Statement function:

1. A simple statement function of one variable is defined to be expression consisting of a predicate symbol and an individual variable.
2. Compound statement function is defined by combining two or more simple statement functions using logical connectives.
3. Astatementfunctionoftwovariablesisanexpressionconsistingofapredicatesymbol and two individual variables.

Example:

1. M(x): x is a mortal.
2. M(x) H(x): where M(x) : x is a man

   H(x): x is a mortal.
3. G(x, y): x is taller than y.

Quantifier:

Quantifier is one which is used to quantify then a ture of variables. There are two Important quantifiers which are for “all” and for “some”, where “some” means “atleast one”.

Universal quantifier:

The universal quantification of P(x) is the statement "P (x) for all values of x in the domain". The notation 6x P(x) or (x) P(x) denotes the universal quantification of P(x). Here6 is called the universal quantifier. We read 6x P(x) as "for all x P(x)" or "for every x P(x)." An element for which P(x) is false is called a counterexample of 6x P(x).

Existential quantifier:

The existential quantification of P(x) is the proposition "There exists an element x in the domain such that P(x)".We use the notation Ex P(x) for the existential quantification of P(x). Here E is called the existential quantifier. The existential quantification Ex P(x) is read as "There is an x such that P(x)", "There is at least one x such that P(x)"or "For some x P(x)".

Universal quantifier	**Existential quantifier**
For all x	For some x
For every x	Some x such that
For each x	There exist an x such that
Every thing x is such that	There is an x such that
Each thing x is such that	There is at least one x such that

Rules of Inference for Quantified Statements	
From 6x A(x) one can conclude A(y)	Universal Specification(instantiation) – RULEUS
From A(x) one can conclude 6y A(y)	Universal Generalization–RULEUG

From Ex A(x) one can conclude A(y)		Existential Specification (Instantiation) – RULEES	
From A(x) one can conclude Ey A(y)		Existential Generalization–RULEEG	
DeMorgan's Laws for Quantifiers			
Negation	Equivalent Statement	When Is Negation True?	When False?
x P(x)	6x P(x)	For every x, P(x) is false.	There is an x for which P(x) is true.
6x P(x)	Ex P(x)	There is an x for which P(x) is false.	P(x) is true for every x.

Problems:

1. Symbolize the statement “Every apple is red”.

   Solution:

   The given statement can be represented as,“For all x,if x is an apple, then x is red” ....(1)

   Let A(x): x is an apple and R(x): x is red.

   Using universal quantifier,(1) can be represented into symbolic form as, (x) (A(x) →R(x)).

2. Symbolize the following statement “All the world loves a lover”.

   Solution:

   The given statement can be represented as, “For all x, if x is a person, then for all y, y is a person and y is a lover, then x loves y”.............. (1)

   Let P(x): x is a person, P(y): y is a person, L(y): y is a lover and R(x,y): x loves y.

   Using universal quantifier, (1) can be represented into symbolic form as,

   (x)(P(x)→((y)((P(y)L(y))→R(x, y)))).

3. What are the negations of the statements 6x(x^2>x) and Ex(x^2=2)?

   Solution:

   The negation of 6x(x^2>x) is the statement (6x(x^2>x)),which is equivalent

To $\exists x(\neg(x^2 > x))$. This can be rewritten as $\exists x(x^2 \leq x)$.

The negation of $\exists x(x^2=2)$ is the statement $\neg(\exists x(x^2=2))$, which is equivalent to $\forall x(\neg(x^2=2))$. This can be rewritten as $\forall x(x^2 \neq 2)$.

The truth values of these statements depend on the domain.

4. What are the negations of the statements "There is an honest politician "and" All Americans eat cheeseburgers"?

Solution:

Let $H(x)$ denote "x is honest". Then the statement "There is an honest politician" is represented by $\exists x\, H(x)$, where the domain consists of all politicians.

The negation of this statement is $\neg(\exists x\, H(x))$, which is equivalent to $\forall x(\neg H(x))$.

This negation can be expressed as "Every politician is dishonest". Let $C(x)$ denote "x eats cheeseburgers". Then the statement "All Americans eat cheeseburgers" is represented by $\forall x(C(x))$, where the domain consists of all Americans.

The negation of this statement is $\neg(\forall x\, C(x))$, which is equivalent to $\exists x(\neg C(x))$.

This negation can be expressed in several different ways, including "Some American does not eat cheeseburgers" and "There is an American who does not eat cheeseburgers".

5. Prove that $\forall x(P(x)\rightarrow Q(x)), \forall x(R(x)\rightarrow \neg Q(x)) \Rightarrow \forall x\,(R(x)\rightarrow \neg P(x))$.

Solution:

Given premises are $\forall x(P(x)\rightarrow Q(x))$ and $\forall x(R(x)\rightarrow\neg Q(x))$.

Conclusion is $\forall x\,(R(x) \rightarrow\neg P(x))$.

{1}	(1)	$\forall x(P(x)\rightarrow Q(x))$	Rule P
{1}	(2)	$P(y)\rightarrow Q(y)$	Rule US, from (1)
{2}	(3)	$\forall x(R(x)\rightarrow \neg Q(x))$	Rule P
{2}	(4)	$R(y)\rightarrow\neg Q(y)$	Rule US, from (3)
{2}	(5)	$Q(y)\rightarrow\neg R(y)$	Rule T, Contra positive law, from(4)
{1, 2}	(6)	$P(y)\rightarrow\neg R(y)$	Rule T, Hypothetical syllogism, from(2)and(5)
{1, 2}	(7)	$R(y)\rightarrow\neg P(y)$	Rule T, Contra positive law, from(6)
{1, 2}	(8)	$\forall x(R(x)\rightarrow \neg P(x))$	Rule UG, from(7)

Therefore $\forall x(R(x)\rightarrow \neg P(x))$ follows from $\forall x\,(P(x)\rightarrow Q(x))$ and $\forall x\,(R(x)\rightarrow \neg Q(x))$.

Nested quantifiers:

Two quantifiers are nested if one is within the scope of the other, such as $\forall x\, \exists y(x + y = 0)$.

Note that everything within the scope of a quantifier can be thought of as a propositional function.

For example, $6x\ Ey\ (x + y = 0)$ is the same thing as $6x\ Q(x)$, where $Q(x)$ is $Ey\ P(x, y)$, where $P(x, y)$ is $x + y = O$. Nested quantifiers commonly occur in mathematics and computer science.

EXAMPLES:

1. Express the statement "some people who trust others are rewarded" in symbolic form.

   Solution:

   Let $P(x)$: x is a person

   $T(x)$: x trusts others

   $R(x)$: x is rewarded

   The symbolic form is

   $(Ex)[P(x) \wedge T(x) \wedge R(x)]$

2. Rewrite the following using quantifiers.

   "Some men are genius".

Solution:

$M(x)$: x is a man

$G(x)$: x is genius

The symbolic form is

$(Ex)\ [M(x) \wedge G(x)]$

3. Let $K(x)$:x is a two wheeler ,$L(x)$:x is a scooter,

   $M(x)$: x is manufactured by Bajaj. Express the following using quantifiers.

   I. Every two wheeler is a scooter

   II. There is a two wheeler that is not manufactured by Bajaj

   III. There is a two – wheeler manufactured by Bajaj that is not a scooter

   IV. Every two –wheeler that is a scooter is manufactured by Bajaj

Solution:

Given $K(x)$: x is a two wheeler

$L(x)$: x is a scooter

$M(x)$:x is manufactured by Bajaj

I. $(\forall x)\ [K(x) \rightarrow L(x)]$

II. $(Ex)\ [K(x) \wedge \neg M(x)]$

III. $(Ex)[K(x) \wedge M(x) \wedge \neg L(x)]$

IV. $(\forall x)[K(x) \rightarrow L(x) \wedge M(x)]$

4. Show that $(x)\ (P(x) \rightarrow Q(x)) \wedge ((x)\ Q(x) \rightarrow R(x)) \Rightarrow (x)\ (P(x) \rightarrow R(x))$

Solution:

	Step	Premises	Rule	Reason
{1}	1	(x)(P(x)→Q(x))Λ ((x)Q(x)→ R(x))	P	Given Premises
	2	(x) (P(x) →Q(x))	T	P ΛQ⇒P
{2}	3	P(y)→Q(y)	Rule US	(x) A(x)⇒A(y)
{1}	4	(x)(Q(x) → R(x))	T	P ΛQ⇒Q
{4}	5	Q(y)→R(y)	US	(x)A(x)⇒A(y)
{3.5}	6	P(y)→R(y)	T	P→Q, Q→R ⇒ P→R
{6}	7	(x)(P(x) → R(x))	UG	A(y)⇒(x)A(x)

5. Prove that ∀x(P(x)→Q(x)),∀x(R(x)→˥Q(x))⇒∀x(R(x)→˥P(x))

Solution:

	Step	Premises	Rule	Reason
	1	(x)(P(x) → Q(x))	P	Given Premises
{1}	2	P(y)→Q(y)	US	(x)A(x)⇒A(y)
	3	(x)(R(x)→ ˥Q(x))	P	Given Premises
{3}	4	R(y)→˥Q(y)	US	(x)A(x)⇒A(y)
{4}	5	Q(y)→˥R(y)	T	P→Q⇒˥Q→˥P
{2.5}	6	P(y)→˥R(y)	T	P→Q, Q→R ⇒P→R
{6}	7	R(y)→˥P(y)	T	P→Q⇒˥Q→˥P
{7}	8	(x) (R(x) → ˥P(x))	US	A(y)⇒(x)A(x)

6. Use indirect method of proof to prove that(∀x)(P(x)VQ(x)) ⇒(∀x)P(x)V (Ex)Q(x)

Solution:

Method 1: method of contradiction: Assume ˥[∀(x) (P(x) V E (x)Q(x)] as an additional premises.

	Step	Premises	Rule	Reason
	1	˥[∀(x)(P(x)V Ex Q(x)]	Additional Premises	
{1}	2	(Ex)˥P(x)Λ(∀x)˥Q(x)	T	Demorgan's
{2}	3	(Ex)˥TP(x)	T	P Λ Q⇒P
{2}	4	(∀x)˥Q(x)	ES	P Λ Q⇒Q
{3}	5	˥P(y)	US	(Ex)A(x)⇒A(y)
{4}	6	˥Q(y)	T	(∀x)A(x)⇒A(y)
{5,6}	7	˥P(y)Λ˥Q(y)	T	P,Q⇒PΛQ

{7}	8	˥[P(y)VQ(y)]	P	Demorgan’s law
	9	(∀x)P(x)VQ(x)	US	Given premises
{9}	10	P(y)VQ(y)	T	(∀x)A(x)⇒A(y)
{8,10}	11	[P(y)VQ(y)]Λ˥[P(y)VQ(y)]	T	P,Q⇒PΛQ
{11}	12	F		P Λ˥P⇔F

Which is false. Therefore by method of contradiction, we have(∀x)(P(x)VQ(x))⇒(∀x)P(x) V Ex Q(x)

Method2: Method of contrapositive

	Step	Premises	Rule	Reason
	1	˥[(x)P(x) VE (x)Q(x)]	Additional Premises	
{1}	2	(Ex)˥P(x)Λ(x)˥Q(x)	T	Demorgan’s law
{2}	3	(Ex)˥P(x)	T	P Λ Q⇒P
{2}	4	(x)˥Q(x)	T	P Λ Q⇒Q
{3}	5	˥P(y)	ES	(Ex)A(x)⇒A(y)
{4}	6	˥Q(y)	US	(x) A(x)⇒A(y)
{5,6}	7	˥P(y)Λ˥Q(y)	T	P,Q⇒PΛQ
{7}	8	˥[P(y)ΛQ(y)]	T	Demorgan’s law
{8}	9	(Ex)˥[P(x)V Q(x)]	EG	A(y)⇒(Ex)A(x)
{9}	10	˥[(x)P(x)VQ(x)]	T	Apply˥

Hence we have

˥[(x)P(x)V(Ex)Q(x)]⇒˥[(x)P(x)VQ(x)]

Therefore,(x)(P(x)VQ(x))⇒(x)(P(x)V(Ex)Q(x))

Translating Mathematical Statements into Statements Involving Nested Quantifiers:

1. Translate the statement "The sum of two positive integers is always positive "into a logical expression.

Solution:

To translate this statement into a logical expression, we first rewrite it so that the implied quantifiers and a domain are shown:"For every two integers, if these integers are both positive, then the sum of these integers is positive." Next, we introduce the variables x and y to obtain "For all positive integers x and y, x + y is positive".

Consequently, we can express this statement as 6x6y ((x>0) □(y >0) →(x + Y >0)), where the domain for both variables consists of all integers.

Note that we could also translate this using the positive integers as the domain. Then the statement "The sum of two positive integers is always positive" becomes "For every two positive integers, the sum of these integers is positive. We can express this as 6x 6y (x+y >0), where the domain for both variables consists of all positive integers.

2. Translate the statement "Every real number except zero has a multiplicative inverse" (A multiplicative inverse of a real number x is a real number y such that x y = 1).

Solution:

The given statement can be rewritten as "For every real number x except zero, x has a multiplicative inverse".

We can rewrite this as" For every real number x ,if xs0,then there exists a real number y such that x y = 1". This can be rewritten as 6x ((xs 0) →Ey (x y = 1)).

3. Express the definition of a limit using quantifiers.

Solution:

The definition of the statement $\lim_{x \to a} f(x)=L$ is: For every real number s>0 there exists a real number $\delta>0$ such that $|f(x)-L| < s$, whenever $0<|x-a|<\delta$.

This definition of a limit can be expressed in terms of quantifiers by $6s\ E\delta\ 6x\ (0 < |x - a| < \delta \rightarrow |f(x) - L| < s)$, where the domain for the variables δ and s consists of all positive real numbers and for x consists of all real numbers.

This definition can also be expressed as $6s > 0\ E\ \delta > 0\ 6\ x\ (0 < |x - a| < \delta \rightarrow |f(x) - L| < s)$ Where the domain for the variables s and δ consists of all real numbers.

Translating English Sentences in to Logical Expressions:

4. Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Solution:

The statement "If a person is female and is a parent, then this person is someone's mother" can be expressed as "For every person x, if person x is female and person x is a parent, then there exists a person y such that person x is the mother of person y".

We introduce the propositional functions $F(x)$ to represent "x is female", $P(x)$ to represent "x is a parent" and $M(x, y)$to represent "x is the mother of y".

The original statement can be represented as $6x\ ((F(x) \wedge P(x)) \rightarrow Ey\ M(x, y))$.

5. Express the statement "Everyone has exactly one best friend" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Solution:

The statement "Everyone has exactly one best friend" can be expressed as "For every person x, person x has exactly one best friend".

Now, introducing the universal quantifier, we see that this statement is the same as "$6x$ (person x has exactly one best friend)",where the domain consists of all people.

To say that x has exactly one best friend means that there is a person y who is the best friend of x, and furthermore, that for every person z, if person z is not the person y, then z is not the best friend of x.

When we introduce the predicate $B(x, y)$ to be the statement "y is the best friend of x" the statement that x has exactly one best friend can be represented as

$$Ey\ (B(x, y) \wedge 6z((z\ s\ y) \rightarrow \neg B(x, z))).$$

It follows that; the given statement can be expressed as

$$6x\ Ey\ (B(x, y) \wedge 6z((z s\ y) \rightarrow \neg B(x, z))).$$

Negating Nested Quantifiers:

6. Express the negation of the statement 6x Ey (x y=1)so that no negation precedes quantifier.

Solution:

By applying De Morgan's laws for quantifiers, we can move the negation in 6x Ey (x y=1) inside all the quantifiers.

Therefore □(6x Ey (x y= 1))eEx (□Ey (x y=1))

eEx 6y (□ (x y=1))

eEx 6y (x ys1).

Hence our negated statement can be expressed as Ex 6y (x ys1).

7. Use quantifiers and predicates to express the fact that $\lim_{x\to a}$ f(x) does not exist.

Solution:

To say that $\lim_{x\to a}$ f(x) does not exist means that for all real numbers L, $\lim_{x\to a}$ f(x)sL.

Since the definition of the statement $\lim_{x\to a}$ f(x)=l is :For every real number s>0 there exists a real number δ>0 suchtha | f(x)– l | <s, whenever 0 < | x– a| <δ.

This definition of a limit can be expressed in terms of quantifiers by

6s >0 Eδ >0 6x (0< |x– a | < δ → | f(x) – l | <s), where the domain for the variables s and δ consists of all real numbers.

m$\lim_{x\to a}$ f(x) s Lcan be expressed as□ (6s>0 Eδ>0 6x (0 < | x– a| < δ → | f(x) – l | < s)).

Successively applying the rules for negating quantified expressions, we construct this sequence of equivalent statements,

□ (6s>0 Eδ>0 6x (0 < |x– a |< δ → | f(x)– l |<s))

Ξ Es >0 (□ (Eδ >0 6x (0 < |x– a |< δ → |f(x) – l |<s))) Ξ

Es>0 6δ > 0 (□ (6x (0< x–|a < δ| → f(x)– l <s|)))

Ξ Es> 0 6δ> 0 Ex (□ (0 < |x– a| < δ → |f(x) – l |<s))

Ξ Es> 0 6δ>0 Ex (□ (□ (0 < | x– a| < δ) v (|f(x) – l |<s)))

Ξ Es >0 6δ >0 Ex (0< |x– a |<δ □ □ (|f(x) – l |<s))

Ξ Es>0 6δ>0 Ex (0 < |x– a |< δ □ |f(x)– l □ s).

Therefore for all real numbers L, $\lim_{x\to a}$ f(x)sLcan be expressed as

6L Es>0 6δ>0 E x (0< x– a < δ □ f(x)–l □ s).

The universe of discourse:

Variables which are quantified stands for only those objects which are members of a particular set or class. Such a set is called the universe of discourse or domain or simply universe.

Example:

Let P(x) : $x < 32$ and Q(x): x is a multiple of 10 with universe of discourse as all positive integers. Find the truth value of $(Ex)(P(x) \rightarrow Q(x))$.

Solution:

Given P(x): $x < 32$ and Q(x) : x is a multiple of 10.

Universe of discourse: $U = \{1, 2, 3, \ldots\}$.

To obtain the truth values of $(Ex)(P(x) \rightarrow Q(x))$.

Let x = 20, 20 ϵU.

Therefore P(20):20< 32 which is true.

Q(20):20isa multipleof10which is true.

Therefore $P(20) \rightarrow Q(20)$ is true.

There exists at least one x=20,$P(x) \rightarrow Q(x)$ is true.

Hence $(Ex)(P(x) \rightarrow Q(x))$ is true.

1.8 Methods of Proof:

Vacuous proof:

A proof that $p \rightarrow q$ is true based on the fact that p is false.

Trivial proof:

A proof that $p \rightarrow q$ is true based on the fact that q is true.

Direct proof:

A proof that $p \rightarrow q$ is true that proceeds by showing that q must be true when p is true.

Proof by contraposition:

A proof that $p \rightarrow q$ is true that proceeds by showing that p must be false when q is false.

Proof by contradiction:

A proof that p is true based on the truth of the conditional statement �p $\rightarrow$q, where q is a contradiction.

Exhaustive proof:

A proof that establishes a result by checking a list of all cases.

Proof by cases:

A proof broken into separate cases where these cases cover all possibilities.

Counter example:

An element x such that p(x) is false.

Uniqueness proof:

A proof that there is exactly one element satisfying a specified property.

Problems:

1. Prove the proposition P(0) where P(n) is the proposition: "If n is a positive integer greater than 1, then $n^2>n$".

Solution:

Let P(0): If 0 is a positive integer greater than 1, then $0^2>0$, since 0 is not a positive integer.

So the proposition is true.

2. Prove that the sum of two odd integers is even.

Solution:

Let us assume that m and n are two odd integers.

Then m=2k+1,where k is any integer and n=2l+1,where l is any integer. Now,

$m + n = 2k + 1 + 2l + 1 = 2(k + l +1)$

Therefore m+n is a multiple of 2.

Hence m + n is even integer.

Therefore the sum of two odd integers is even.

3. Prove that $\sqrt{2}$ is irrational.

Solution:

We prove this result using contradiction method(indirect method).

Suppose that $\sqrt{2}$ is rational.

Then $\sqrt{2} = \frac{p}{q}$, qs0, where p and q are integers, they have no common divisors.

Squaring on both sides, we have $2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2$.

Therefore p^2 is an even integer (since $2q^2$ is a multiple of 2), implies p is also an even integer.

Let us take $p=2m$, where m is an integer.

Then $(2m)^2 = 2q^2 \Rightarrow q^2=2m^2$.

Therefore q^2 is an even integer, implies q is also an even integer.

Let $p=2n$, where n is an integer.

Then p and q have common divisor 2.

Therefore this is a contradiction to our assumption p and q have no common divisors.

By the method of contradiction, $\sqrt{2}$ is irrational.

4. Prove that the sum of an irrational and rational number is irrational.

Solution:

We prove this result using contradiction method (indirect method).

Suppose that the sum of an irrational and rational number is rational.

Then $a = b + c$, where a, c are rational and b is irrational.

$\therefore b=a-c=a+(-c)$ is rational, since the sum of two rational numbers is also rational.

Therefore, this is a contradiction to our assumption that a is rational.

By the method of contradiction, a is irrational.

Hence the sum of an irrational and rational number is irrational.

5. Prove that if n is an integer and n^3+5 is odd, then n is even.

Solution:

Given n is an integer and n^3+5 is odd.

To prove n is even.

Now, we prove this result using contraposition method (indirect method).

Let us assume that n is odd.

To prove n^3+5 is even.

Let $n=2k+1$, where k is any integer.

Therefore $n^3+5=(2k+1)^3+5=(2k)^3+3(2k)^2(1)+3(2k)(1)^2+(1)^3+5$

$$=8k^3+12k^2+6k+6$$

$$n^3+5=2(4k^3+6k^2+3k+3)$$

Therefore $n^3 + 5$ is an even integer, since $2(4k^3 + 6k^2 + 3k + 3)$ is a multiple of 2.

By the method of contra positive, if n is an integer and n^3+5 is odd, then n is even.

6. Show that at least four of any 22 days must fall on the same day of the week.

Solution:

Let p be the proposition "At least four of 22 chosen days fall on the same day of the week". Suppose that $\neg p$ is true. This means that at most three of the 22 days fall on the same day of the week.

Because there are seven days of the week, this implies that at most 21 days could have been chosen because for each of the days of the week, at most three of the chosen days could fall on that day.

This contradicts the hypothesis that we have 22 days under consideration.

That is, if r is the statement that 22days are chosen, then we have shown that $\neg p \rightarrow (r \wedge \neg r)$.

Consequently, we know that p is true. We have proved that at least four of 22 chosen days fall on the same day of the week.

UNIT-II

PREDICATE CALCULUS

2.0 INTRODUCTION

So far, our discussion of symbolic logical has been limited to the consideration of statements and statement formulas. The inference theory was also restricted in the sense that the premises and conclusions were all statement .The symbols $p, q, r, \ldots p_1, q_1$ …..were used for statement or statement variables. The statements were taken as basic units of statement calculus , and no analysis of any atomic statement was admitted. Only compound formulas were analyzed , and this analysis was done by studying the forms of the compound formulas , that is, the connections between the constituent atomic statements.

It was not possible to express the fact that any two atomic statements have some features in common. In order to investigate questions of this nature , we introduce the concept of a predicate in an atomic statement. The logic based on the analysis of predicates in any statement is called predicate logic.

2.1 Predicates

Consider the following two statements:

[i] Radha is a girl. [2.1]

[ii] Seeta is a girl. [2.2]

Here, the first part Radha or Seeta is the subject of the statement. The second part "is a girl'', Which refers to a property that the subject can have , is called the predicate. We symbolize a predicate by a capital letter and the names of individuals or objects in general by lower case letters. Here, the predicate "is a girl'' is denoted by G, Radha by r and Seeta by s . Hence, the statement "Radha is a girl'' is denoted by G [r], While "Seeta is a girl'' is denoted by G[s].In general, any statement of the type "P is Q'', Where Q is a predicate and p is the subject, can be denoted by Q[p]. A predicate requiring m [m > 0]

Names is called an m-place predicate. Equations [2.1] and [2.2] are 1-place predicates. A statement is called a 0 – place predicate when no names are associated with the statement.

Example 2.1

Provide examples of predicates.

Solution:

Consider the statements

[i] Radha is taller than Seeta. [2.3]

If "T'' denotes "is taller than", then T is a two-place predicate. [2.4]

[ii] "Rita sits between Nita and Sita" is a three – place predicate.

[iii] "Green and Miller played bridge against John and Smith" is a four – place predicate.

2.1.2 Statement Function, Variables and Quantifiers

Consider the following statements:

*Somu is mortal.

*India is mortal.

*A table is mortal.

Let H be the predicate ‘ is mortal’; s be the name Somu ; i be the name ‘India’ and t be the table. Then, H[s] , H[i]and H[t] denote the above statements. If we write H[x] for “x is mortal”, then H[s], H[i]and H[t] can be obtained from H[x] by replacing x by an appropriate name.

Note that H[x] is not a statement, but when x is replaced by the name of an object, it becomes a statement.

*A simple statement function of one variable is defined as an expression consisting of a predicate symbol and an individual variable.

*A statement function becomes a statement when the variable is replaced by the name of any object . The statement resulting from a replacement is called a substitution instance of the statement functions and the logical connectives.

Example 2.2

Provide some examples of compound statements.

Solution :

$M(x) \wedge H(x), M(x) \rightarrow H(x), \neg H(x), M(x) \vee \neg H(x)$, ect., are all examples of compound statement functions, where $M(x)$ represent “x is a man” and $H(x)$ represent “x is a mortal” .

An extension of this idea to the statement functions of two or more variables is straightforward.

Example 2.3

Provide examples of statement functions.

Solution:

Consider the statements function of two variables:

$G(x, y)$: x is taller than y.

If both x and y are replaced by the names of objects, we get a statement.

If m represents ‘Mr.Miller’ and f represent ‘Mr.Fox’, then we have

$G(m, f)$: Mr.Miller is taller than Mr.Fox and

$G(f, m)$: Mr.Fox is taller than Mr.Miller

It is possible to form statement functions of two variable by using statement functions of one variable.

Example 2.4

Form a compound statement from the following:

[i] $M(x)$: x is a man

[ii] $H(y)$: y is a mortal

Solution

Given $M(x)$: x is a man

$H(y)$: y is a mortal

Then we write

$M(x) \wedge H(y)$: x is a man and y is mortal

Note 2.1 It is not possible to write every statement function of two variables using statement functions of one variable . One way of obtaining statements from any statement function is to replace the variable by the names of the object.

Example 2.5 :

Let $P(x)$ denote the statement ' $x > 3$ '. What are the truth values of $P(4)$ and $P(2)$?

Solution :

The statement $P(4)$ can be obtained by setting $x = 4$ in the statement $x > 3$. Hence, $P(4)$ is the statement ' $4 > 3$ ', Which is true. Similarly, $P(2)$ is the statement ' $2 > 3$ ', Which is false.

Hence, the truth values of $P(4)$ and $P(2)$ are T and F, respectively.

Example 2.6 :

Let $Q(x, y)$ denote the statement ' $x = y + 3$ '. What are the truth values of the propositions $Q(1,2)$ and $Q(3,0)$?

Solution :

$Q(x, y)$: $x = y + 3$.

The statement $Q(1,2)$ can be obtained by putting $x = 1$ and $y = 2$ in the statement $Q(x, y)$.

Hence, $Q(x, y)$ is the statement ' $1 = 2 + 3$ ', Which is false.

The statement $Q(3,0)$ is the statement ' $3 = 0 + 3$ ', Which is true.

Thus, the truth values of $Q(1,2)$ and $Q(3,0)$ are F and T, respectively.

2.1.3 Quantifiers

There are two types of quantifiers. They are the following:

[i] Universal quantifiers

[ii] Existential quantifiers

Universal quantifiers

Consider the following statements:

*All men are mortal.

*Every apple is red.

*An integer is either positive or negative.

These statements can be written as

For all x, if x is a man , then x is mortal.

For all x, if x is an apple , then x is red.

For all x, if x is an integer , then x is either positive or negative.

These statements can be symbolized as

$(x)\ (M\ (x) \to H\ (x)); (x) (\ A(x) \to R\ (x))\ \textit{and}\ (x) (Z\ (x) \to (P\ (x)\ \mathrm{v}\ N\ (x))),$

$M\ (x)$: x is a man; $H\ (x)$: x is mortal

$A(x)$: x is an apple; $R\ (x)$: x is red

$Z\ (x)$: x is an integer; $P\ (x)$: x is positive; $N\ (x)$: x is negative.

We symbolize 'for all x' by the symbol $(6x)$ or (x) and it is called universal quantifier.

The universal quantification of $P\ (x)$ is the proposition $P\ (x)$ that is true for all values of x in the univers of discourse.

The notation ' $(6x)\ P\ (x)$ 'denotes the universal quantification of $P\ (x)$. The proposition $6x\, P\ (x)$ is also expressed as 'for all x $P\ (x)$ ' or 'for every x $P\ (x)$ '.

Existential quantifiers

Consider the following statements:

*There exists a man.

*Some men are clever.

*Some real numbers are rational.

These statements can be rewritten as

*There exists an x such that x is a man.

* There exists an x such that x is a man and x is clever.

* There exists an x such that x is a real number and x is a rational number.

The above statements can be symbolized as

$(\mathrm{E}x)\, M\ (x) : (\mathrm{E}x) (M\ (x)\ \hat{}\ (Cx))\ \textit{and}\ (\mathrm{E}x) (R_1\ (x)\ \mathrm{v}\ R_2\ (x))$, respectively.

Where $M\ (x)$: x is a man ; $C\ (x)$: x is clever

$R_1\ (x)$: x is a real number and $R_2\ (x)$: x is a rational number.

Existential quantifier is denoted by the symbol [Ex] and stands for 'there is at least one x such that 'or 'there exists an x such that ' or "for some x ".

The existential quantification of $P\ (x)$ is the proposition "there exists an element ' x ' in the universe of discourse such that $P\ (x)$ is true".

We use the notation $\text{E}x\ P(x)$ for the existential quantification; here, ' E 'is called the existential quantifier.

Example 2.7 Let $P(x)$ be the statement ' $x+1>x$ '.What is the truth value of the quantification $\text{E}x\ P(x)$,where the universe of discourse consists of all real numbers?

Solution $P(x)$: $x+1>x$.

Since $P(x)$ is true for all real numbers x , the quantification $\text{E}x\ P(x)$ is true.

Example 2.8 What is the truth value of the following quantifications?

[i] $(6x)\ Q(x)$, Where $Q(x)$ is the statement' $x<2$ ' and the universe of discourse consists of all real numbers?

[ii] $(6x)\ P(x)$, Where $P(x)$ is the statement' $x^2<10$ ' and the universe of discourse consists of the positive integers not exceeding 4 ?

Solution :

[i] $Q(x)$: $x<2$.

$Q(x)$ is not true for every real number x, since $Q(3)$ is false.

Thus, $(6x)\ Q(x)$ is false.

[ii] The statement $(6x)\ P(x)$ is $P(1)\ \hat{}\ P(2)\ \hat{}\ P(3)\ \hat{}\ P(4)$, since the universe of discourse is $\{1,2,3,4\}$.

$P(4)$ is the statement $4^2<10$, Which is not true.

Hence, $(6x)\ P(x)$ is false.

Example 2.9 :

What is the truth value of the following quantifications?

[i] $\text{E}x\ P(x)$, Where $P(x)$ denotes the statement ' $x>3$ ' and the universe of discourse consists of all real numbers?

[ii] $\text{E}x\ Q(x)$, Where $Q(x)$ denotes the statement ' $x=x+1$ ' and the universe of discourse is the set of all real numbers?

Solution :

[i] $P(x)$: $x>3$.

When $x=4$, $4>3$,which is ture.

Hence, when $x=4$, $\text{E}x\ P(x)$ is ture.

[ii] $Q(x)$: $x=x+1$.

Since $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$ is false, I.e., $\mathrm{E}x\, Q(x)$ is false.

Example 2.10

Express the statement "Every student in the class has studied calculus" as a universal quantification.

Solution :

Let $P(x)$ denote the statement 'x has student calculus'. Then, the given statement can be written as $(6x)\ P(x)$, where the universe of discourse consists of the students in the class. This statement can also be expressed as $6x\,(S(x) \rightarrow P(x))$, where $S(x)$ is the statement 'x is the student in this class ', $P(x)$ is the statement 'x has studied calculus' and the universe of discourse is the set of all students.

Table 2.1 shows the quantifier of Example 2.10.

Table 2.1 Quantifier for Example 2.10

Statement	When ture?	When false?
$(x)\ P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\mathrm{E}x\, P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Example 2.11

Write each of the following in symbolic form:

[i] All men are giants.

[ii] No men are giants.

[iii] Some men are giants.

[iv] Some men are not giants.

Solution:

Assume that the universe of discourse consists of objects for which some are not men. Let

$M(x)$: x is a man

$G(x)$: x is a giant

[i] The given statement means "for all x, if x is a man, then x is a giant". Thus , the symbolic form is

$(x)\ (M(x) \rightarrow G(x))$.

[ii] The symbolic form is $(x)\ (M(x) \rightarrow \neg G(x))$.

[iii] The symbolic form is $(\mathrm{E}x)(M(x) \hat{}\, G(x))$.

[iv] The symbolic form is $(\mathrm{E}x)(M(x) \hat{}\, \neg G(x))$.

Example 2.12

Let $P(x)$: x is a person

$F(x, y)$: x is the father of y

$M(x, y)$: x is the mother of y

Write the predicate ‘ x is the father of the mother f y’.

Solution :

We assume that a person ‘z’ exists such that x is the father of z and z is the mother of y.

Therefore, the given predicate is symbolized as

$$(\mathrm{E}x)\left(P(z) \wedge F(x, z) \wedge M(z, y)\right).$$

Example 2.13

Symbolize the expression ‘All the world loves a lover’.

Solution :

Note that the given expression means that ‘ everybody loves a lover’.

Let $P(x)$: x is a person

$L(x)$: x is a lover

$R(x, y)$: x loves y

The required symbolic expression is

$$(x)\left(P(x) \rightarrow (y)\left[\{P(y) \wedge L(y)\} \rightarrow R(x, y)\right]\right)$$

Universe of Discourse

We restrict the universe by limiting the class of individuals or objects under consideration. Then the variables that are quantified stand for only those object that are members of a particular set or class. Such a restricted class is called the universe of discourse or the domain of individuals or simply the universe.

Example : 2.14

Symbolize the statement ‘All men are giants‘.

Solution :

Let $G(x)$: x is a giant

$M(x)$: x is a man

The given statement can be symbolized as $(x)\left(M(x) \rightarrow G(x)\right)$.

But when we restrict the variable x to the universe that is the class of men, then the given statement can be symbolized as $(x)\ G(x)$.

Example : 2.15

Consider the statement ‘ Given any positive integer, there is a greater positive integer”.

Symbolized this statement by both using and not using the set of positive integers as the universe of discourse.

Solution : Let the variable x and y be restricted to the set of positive integers.

Let $G(x, y)$: x is greater than y

Thus, the given statement can be symbolized as $(x)\,(\mathrm{E}y)\,G(x,y)$.

Suppose there is no restriction on the variables.

Let $P(x)$: x is a positive integer.

The given statement can be symbolized as

$$(x)\,\big(P(x) \rightarrow (\mathrm{E}y)\big(P(y) \wedge G(y,x)\big)\big).$$

Example : 2.16

Express the negation of the following statement using quantifiers and English.

[i] If the teacher is absent, then some students do not keep quiet.

[ii] All the students keep quiet and the teacher is present.

[iii] Some of the students do not keep quiet or the teacher is absent.

[iv] No one has done every problem in the exercise.

Solution :

[i] Let T represent the presence of the teacher and $Q(x)$ represent ' x keeps quiet '.

Then the given statement is $\neg T \rightarrow (Fx)\neg Q(x)$

$$\Leftrightarrow \neg T \rightarrow \neg(x)Q(x)$$
$$\Leftrightarrow T \vee \neg(x)Q(x)$$

Negation of the given statement is

$$\neg\big(T \vee \neg(x)Q(x)\big) \Leftrightarrow \neg T \wedge (x)Q(x)$$

All the students keep quiet and the teacher is absent.

[ii] The given statement is $(x)Q(x) \wedge T$

Negation of the given statement is

$$\neg\left[(x)Q(x) \wedge T\right] \Leftrightarrow \neg\left[(x)Q(x)\right] \vee [\neg T] \Leftrightarrow$$
$$(\mathrm{E}x)\neg Q(x) \vee \neg T$$

i.e., Some of the students do not keep quiet or the teacher is absent.

[iii] The given statement is $(\mathrm{E}x)\neg Q(x) \vee \neg T$

$$\Leftrightarrow \neg\big((x)Q(x) \vee \neg T\big)$$

Negation of the given statement is

$$\neg\big(\neg(x)Q(x) \vee \neg T\big)$$
$$\Leftrightarrow (x)Q(x) \wedge T$$

i.e.,All the students keep quiet and the teacher is absent

[iv] Let $D(x,y)$ represent ' x done the problem '.

The given statement is $\neg(\mathrm{E}x)((y) D(x, y))$

Negation of the given statement is

$\neg\neg(\mathrm{E}x)((y) D(x, y))$

e $(\mathrm{E}x(y) D(x, y))$

i.e., There are students solved every problem in the exercise.

Free and bound variables

Let x be variable and D be a set. $R(x)$ is a proposition. Then $R(x)$ is called a propositional function or predicate with respect to the domain D if for each value of x in D, $R(x)$ is a proposition. That is, $R(x)$ is true or false. D is called the domain of the discourse and x is called the free variable.

The variable x appearing in $6x$ ⍰ $R(x)$ or $\mathrm{E}x\, R(x)$ is called a bound variable. [It is bounded by quantifiers 6 and E].

If $R(x, y)$ is a sentence, then in the sentence $6x \quad R(x, y)$ the variable x is bounded.

In other words, when a quantifier is used on the variable x or when we assign a value to this variable, then the occurrence of the variable is bound. An occurrence of a variable that is not bounded by a quantifier or set equal to a particular value is said to be free.

Differences between Free Variables and Bound Variables

Aspect	Free Variable	Bound Variable
Definition	A variable not constrained by a quantifier or operator.	A variable that is constrained by a quantifier or operator.
Scope	Can take any value within its domain.	Limited to specific values or conditions defined by the context.
Examples in Logic	In ∀x (x + y = z), y and z are free.	In ∀x (x + y = z), x is bound by ∀.
Examples in Summation	Not applicable as free variables are not part of the summation limits.	In ∑ni=1i, i is bound by the summation limits 1 to n.
Examples in Integration	Not applicable as free variables are not part of the integration limits.	In ∫21 x2 dx, x is bound by the integration limits 0 and 1.
Examples in Functions	In f(x) = $x^2 + c$, c is free.	In f(x) = x^2 +c, x is bound as it is the argument of the function.
Role in Expressions	Represents parameters that can be manipulated or solved for.	Represents variables within the scope of operations like summation, integration, or quantification.
Flexibility	More flexible, as they can be any value within their domain.	Less flexible, as their values are constrained by the expression's context.

Importance	Crucial for forming general expressions and broad applicability.	Essential for defining the limits and conditions of mathematical and logical operations.

2.2 INFERENCE THEORY OF PREDICATE CALCULUS

To understand the inference theory of predicate calculus, it is important to be familiar with the following rules: The rules of universal of a logical consequence in predicate logic, we introduce four more additional rules of inference to the rules of inferences already presented in our discussion on propositional logic.

Rule US : Universal specification or instantiation

$$(x)\,A(x) \Rightarrow A(y)$$

From $(x)\,A(x)$, one can conclude $A(y)$.

Rule ES : Existential specification

$$(\mathrm{E}x)\,A(x) \Rightarrow A(y)$$

From $(\mathrm{E}x)\,A(x)$,one can conclude $A(y)$ provided that y is not free in any given premise and is also not free in any prior step of the derivation.

Rule EG : Existential generalization

$$A(x) \Rightarrow (\mathrm{E}y)\,A(y)$$

From $(x)\,A(x)$, one can conclude $(\mathrm{E}y)\,A(y)$.

Rule UG : Universal generalization

$$A(x) \Rightarrow (y)\,A(y)$$

From $A(x)$, one can conclude $(y)\,A(y)$ provided y is not free in any of the given premises and provided that if x is free in a prior step that resulted from the use of ES, then no veriables introduced by that use of ES appear free in $A(x)$.

Equivalence formulas

$$E_{31} : (\exists x)\left[A(x) \vee B(x)\right] \Leftrightarrow (\exists x)\, A(x) \vee (\exists x)\, B(x)$$

$$E_{32} : (x)\big(A(x) \wedge (\exists x)\, B(x)\big) \Leftrightarrow (x)\, A(x) \wedge (x)\, B(x)$$

$$E_{33} : \neg(\exists x)\, A(x) \Leftrightarrow (x)\neg A(x)$$

$$E_{34} : \neg(x)\, A(x) \Leftrightarrow (\exists x)\neg A(x)$$

$$E_{35} : (x)\big(A \vee B(x)\big) \Leftrightarrow A \vee (x)\, B(x)$$

$$E_{36} : (\exists x)\big(A \wedge B(x)\big) \Leftrightarrow A \wedge (\exists x)\, B(x)$$

$$E_{37} : (x)\, A(x) \rightarrow B \Leftrightarrow (\exists x)\big(A(x) \rightarrow B\big)$$

$$E_{38} : (\exists x)\, A(x) \rightarrow B \Leftrightarrow (x)\big(A(x) \rightarrow B\big)$$

$$E_{39} : A \rightarrow (x)\, B(x) \Leftrightarrow (x)\big(A \rightarrow B(x)\big)$$

$$E_{40} : A \rightarrow (\exists x)\, B(x) \Leftrightarrow (\exists x)\big(A \rightarrow B(x)\big)$$

$$E_{41} : (\exists x)\big(A(x) \rightarrow B(x)\big) \Leftrightarrow (x)\, A(x) \rightarrow (\exists x)\, B(x)$$

$$E_{42} : (\exists x)\, A(x) \rightarrow (x)\, B(x) \Leftrightarrow (x)\big(A(x) \rightarrow B(x)\big)$$

Implication formulas

$$I_{15} : (x)\, A(x) \vee (x)\, B(x) \Rightarrow (x)\big(A(x) \vee B(x)\big)$$

$$I_{16} : (\exists x)\big(A(x) \wedge B(x)\big) \Rightarrow (x\exists)\, A(x) \wedge (x\exists)\, B(x)$$

Validity of arguments

In Mathematics, an argument or a proof of a theorem consists of a finite sequence of statements ending in a conclusion.

A finite sequence $p_1, p_2, \ldots\ldots. p_{n-1}, p_n$ of statements is called an argument. The final statement p_n is the conclusion and the statements $p_1, p_2, \ldots\ldots. p_{n-1}$ are called the premises of the argument.

An argument form is called valid if whenever all the hypothesis are true, the conclusion is also true.

An argument $p_1, p_2, \ldots\ldots. p_{n-1}, p_n$ is called logically valid if the statement formula $\left(p_1 \wedge p_2 \wedge \ldots\ldots, \wedge p_{n-1}\right) \rightarrow p_n$ is a tautology.

When all propositions used in a valid argument are true, it leads to a correct conclusion. If one or more false propositions are used then a valid argument can lead to an incorrect conclusion.

Sometimes we write an argument in the following form

$$
\begin{array}{c}
p_1 \\
p_2 \\
. \\
. \\
. \\
. \\
p_{n-1} \\
p_n
\end{array}
$$

To test the logical validity of an argument written in a natural language, first we write each of the premises and conclusions with the help of statement notation and logical connectives. Then we cheek whether

$p_1 \wedge p_2 \wedge \ldots\ldots, \wedge p_{n-1}$ logically implies p_n.

Example 2.17. :

Verify the validity of the following argument:

"All men are mortal . Socrates is a man. Therefore, Socrates is mortal".

Or

Show that $(x)\left[H(x)\to M(x)\right] \wedge H(s) \Rightarrow M(s)$.

Solution :

Let us represent the statement as follows:

$H(x)$: x is a man

$M(x)$: x is mortal

S : Socrates

Thus, We have to show that $(x)\left[H(x)\to M(x)\right] \wedge H(s) \Rightarrow M(s)$.

$\{1\}\ (1)\ (x)\left[H(x)\to M(x)\right]$	*rule P*
$\{1\}\ (2)\ H(s)\to M(s)$	*rule US*, (1)
$\{3\}\ (3)\ H(s)$	*rule P*
$\{1,3\}\ (4)\ M(s)$	*rule T*, $(2),(3)$ *and* $p, p\to q \Rightarrow q$

Hence , the result.

Note : 2.2: The problem discussed in Example 2.17 is a symbolic translation of a well – known argument known as the Socrates argument.

Example 2.18 :

Verify the validity of the following argument: "Every living thing is a plant or an animal. Joe's goldfish is alive and it is not a plant. All animals have hearts. Therefore, Joe's goldfish has a heart".

Solution :

Let us assume that the universe of discourse is all living things.

Let $P(x)$: x is a plant

$A(x)$: x is an animal

$H(x)$: x has a heart

f : Joe's goldfish

Then the given statement becomes

$$(x)\big(P(x) \vee A(x)\big), \neg P(f), (x)\big(A(x) \rightarrow H(x)\big) \Rightarrow H(f).$$

$\{1\}$	$(1)\ (x)\big(P(x) \vee A(x)\big)$	$rule\ \ P$
$\{1\}$	$(2)\ P(f) \vee A(f)$	$rule\ \ US, (1)$
$\{1\}$	$(3)\ \neg P(f) \rightarrow A(f)$	$rule\ \ T, (2)\ and\ 'p \rightarrow q \Leftrightarrow \neg p \vee q'$
$\{4\}$	$(4)\ \neg P(f)$	$rule\ \ P$
$\{1,4\}$	$(5)\ A(f)$	$rule\ \ T, (3), (4)\ and\ 'p, p \rightarrow q \Rightarrow q'$
$\{6\}$	$(6)\ (x)\big(A(x) \rightarrow H(x)\big)$	$rule\ \ P$
$\{6\}$	$(7)\ A(f) \rightarrow H(f)$	$rule\ \ US, (6)$
$\{1,4,6\}$	$(8)\ H(f)$	$rule\ \ T, (5), (7)\ and\ 'p, p \rightarrow q \Rightarrow q'$

Hence, the given statement is valid.

Example 2.19 :

Establish the validity of the following argument : "All integers are rational numbers. Some rational numbers are powers of 2".

Solution :

Let $P(x)$: x is an integer

$R(x)$: x is an animal

$S(x)$: x is a power of 2

f : Joe's goldfish

Hence the given statement becomes

$$(x)\big(P(x) \rightarrow R(x)\big), (\mathrm{E}x)\big(P(x) \wedge S(x)\big) \Rightarrow (\mathrm{E}x)\big(R(x) \wedge S(x)\big).$$

$\{1\}$	(1)	$(\exists x)(P(x) \wedge A(x))$	rule P
$\{1\}$	(2)	$P(c) \wedge S(c)$	rule $ES,(1)$
$\{1\}$	(3)	$P(c)$	rule $T,(2)$ and $'p \wedge q \Rightarrow p'$
$\{1\}$	(4)	$S(c)$	rule $T,(2)$ and $'p \wedge q \Rightarrow q'$
$\{5\}$	(5)	$(x)(P(x) \to R(x))$	rule P
$\{5\}$	(6)	$P(c) \to R(c)$	rule $US,(5)$
$\{1,5\}$	(7)	$R(c)$	rule $T,(3),(6)$ and $'p, p \to q \Rightarrow q'$
$\{1,5\}$	(8)	$R(c) \wedge S(c)$	rule $T,(4),(7)$ and $'p, q \Rightarrow p \wedge q'$
$\{1,5\}$	(8)	$(\exists x)(R(x) \wedge S(x))$	rule $EG,(8)$

Hence, the given statement is valid.

Example 2.20. :

Show that $(x)[P(x) \to Q(x)] \wedge (x)[Q(x) \to R(x)] \Rightarrow (x)[P(x) \to Q(x)]$.

Solution :

$\{1\}$	(1)	$(x)(P(x) \to Q(x))$	rule P
$\{1\}$	(2)	$P(y) \to Q(y)$	rule $US,(1)$
$\{3\}$	(3)	$(x)(Q(x) \to R(x))$	rule P
$\{3\}$	(4)	$Q(y) \to R(y)$	rule $US,(3)$
$\{1,3\}$	(5)	$P(y) \to R(y)$	rule $T,(2),(4)$ and $p \to q, q \to r \Rightarrow p \to r'$
$\{1,3\}$	(6)	$(x)(P(x) \to R(x))$	rule $UG,(5)$

Example 2.21. :

Show that $(\exists x)M(x)$ follows logically from the premises $(x)[H(x) \to M(x)]$ and $(\exists x)H(x)$.

Solution :

$\{1\}$	(1)	$(\exists x)H(x)$	rule P
$\{1\}$	(2)	$H(y)$	rule ES
$\{3\}$	(3)	$(x)(H(x) \to M(x))$	rule P
$\{3\}$	(4)	$H(y) \to M(y)$	rule $US,(3)$
$\{1,3\}$	(5)	$M(y)$	rule $T,(2),(4)$ and $p, p \to q \Rightarrow q'$
$\{1,3\}$	(6)	$(\exists x)M(x)$	rule $EG,(5)$

Example 2.22. :

Show that $(\exists x)\left[P(x) \wedge Q(x)\right] \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$.

Solution :

$\{1\}$	(1)	$(\exists x)(P(x) \wedge Q(x))$	*rule P*
$\{1\}$	(2)	$P(y) \wedge Q(y)$	*rule ES*, (1)
$\{1\}$	(3)	$P(y)$	*rule T*, (2) *and* $p \wedge q \Rightarrow p'$
$\{1\}$	(4)	$(\exists x)P(x)$	*rule EG*, (3)
$\{1\}$	(5)	$Q(y)$	*rule T*, (2) *and* $p \wedge q \Rightarrow q'$
$\{1\}$	(6)	$(\exists x)Q(x)$	*rule EG*, (4)
$\{1\}$	(7)	$(\exists x)P(x) \wedge (\exists x)Q(x)$	*rule T*, $(4), (5)$ *and* $'p, q \Rightarrow p \wedge q'$

Note 2.3 is the converse true?

$\{1\}$	(1)	$(\exists x)P(x) \wedge (\exists x)Q(x)$	*rule P*
$\{1\}$	(2)	$(\exists x)P(x)$	*rule T*, (1) *and* $p \wedge q \Rightarrow p'$
$\{1\}$	(3)	$(\exists x)Q(x)$	*rule T*, (1) *and* $p \wedge q \Rightarrow q'$
$\{1\}$	(4)	$P(y)$	*rule ES*, (2)
$\{1\}$	(5)	$Q(s)$	*rule ES*, (3)

Here in step (4) , y is fixed , and it is not possible to use that variable again in step (5). Hence, the converse is not true.

Example 2.23. :

Show that from $(\exists x)(F(x) \wedge S(x)) \rightarrow (y)(M(y) \rightarrow W(y))$ and $(\exists y)\left[M(y) \wedge \neg W(y)\right]$ the conclusion $(x)(F(x) \rightarrow \neg S(x))$ follows.

Solution :

$\{1\}$	(1)	$(\exists y)\left(M(y) \to \neg W(y)\right)$	*rule P*
$\{1\}$	(2)	$M(z) \wedge \neg W(z)$	*rule ES* (1)
$\{1\}$	(3)	$\neg\left(M(z) \to \neg W(z)\right)$	*rule T*, (2) *and* $\neg(p \to q) \Leftrightarrow p \wedge \neg q'$
$\{1\}$	(4)	$(\exists y)\left(M(y) \to W(y)\right)$	*rule EG*, (3)
$\{1\}$	(5)	$\neg(y)\left(M(y) \to W(y)\right)$	*rule T*, (4) *and* $\neg(x)A(x) \Leftrightarrow (\exists x)A(x)'$
$\{1\}$	(6)	$(\exists x)\left(F(x) \wedge S(x)\right) \to$ $(y)\left(M(y) \to W(y)\right)$	*rule EG*, (5)
$\{1,6\}$	(7)	$\neg(\exists x)\left(F(x) \wedge S(x)\right)$	*rule T*, (5), (6) *and* $\neg q, p \to q \Rightarrow \neg p'$
$\{1,6\}$	(8)	$(x)\neg\left(F(x) \wedge S(x)\right)$	*rule T*, (7) *and* $\neg(\exists x)A(x) \Leftrightarrow (x)\neg A(x)'$
$\{1,6\}$	(9)	$\neg\left(F(z) \wedge S(z)\right)$	*rule US*, (8)
$\{1,6\}$	(10)	$\neg F(z) \vee \neg S(z)$	*rule T*, (9) *and by De Morgan's laws*
$\{1,6\}$	(11)	$F(z) \to \neg S(z)$	*rule T*, (10) *and* $p \to q \Rightarrow \neg p \vee q'$
$\{1,6\}$	(12)	$(x)\left(F(x) \to \neg S(x)\right)$	*rule UG*, (11)

Example 2.24. :

Show that $(x)\left(P(x) \vee Q(x)\right) \Rightarrow (x)P(x) \vee (\exists x)Q(x)$.

Solution :

We shall use the indirect method of proof. We assume $\neg\left((x)P(x) \vee (\exists x)Q(x)\right)$ as an additional premise and prove a contradiction.

$\{1\}$	(1)	$\neg\left((x)P(x) \vee (\exists x)Q(x)\right)$	*rule P* (*additional premise*)
$\{1\}$	(2)	$\neg(x)P(x) \wedge \neg(\exists x)Q(x)$	*rule T*, (1) *and by De Morgan's laws*
$\{1\}$	(3)	$\neg(x)P(x)$	*rule T*, (2) *and* $'p \wedge q \Rightarrow p'$
$\{1\}$	(4)	$(\exists x)\neg P(x)$	*rule T*, (3) *and* $'\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)'$
$\{1\}$	(5)	$\neg(\exists x)P(x)$	*rule T*, (2) *and* $'p \wedge q \Rightarrow q'$
$\{1\}$	(6)	$(x)\neg P(x)$	*rule T*, (5) *and* $'\neg(\exists x)A(x) \Leftrightarrow (x)\neg A(x)'$
$\{1\}$	(7)	$\neg P(y)$	*rule ES*, (4)
$\{1\}$	(8)	$\neg Q(y)$	*rule US*, (6)
$\{1\}$	(9)	$\neg P(y) \wedge \neg Q(y)$	*rule T*, (7), (8) *and* $'p, q \Rightarrow p \wedge q'$
$\{1\}$	(10)	$\neg\left(P(y) \vee Q(y)\right)$	*rule T*, (8) *and by De Morgan's laws*
$\{1\}$	(11)	$(x)\left(P(x) \vee Q(x)\right)$	*rule P*
$\{1\}$	(12)	$P(y) \vee Q(y)$	*rule UG*, (11)
$\{1\}$	(13)	$\neg\left(P(y) \vee Q(y)\right) \wedge$ $P(y) \vee Q(y)$	*rule T*, (10), (12) *and* $'p, q \Rightarrow p \wedge q'$
$\{1\}$	(14)	F	*rule T and* (13)

This is contradiction. Hence the statement is valid.

NESTED QUANTIFIERS

Nested Quantifiers are quantifiers that occur within the scope of other quantifiers, such as in the statement $\forall m\ \exists n\ (m + n = 0)$ .Mostly, nested quantifiers occur in mathematics and computer science. Sometimes, nested quantifiers are very difficult to understand. In such a case, we can use the rules that we have already discussed in the earlier section.

Translating Statements Involving Nested Quantifiers

Complicated expressions involving quantifiers occur in many contexts. To understand the statements involving many quantifiers, first write out what the quantifiers and predicates in the expression mean and then express this meaning in a simple sentence.

Example: 1

Translate the following statements into English, where the universe of discourse for each variable consists of all real numbers.

$(a)\ \ \forall m\ \forall n\,(m + n = n + m)$

$(b)\ \ \forall m\ \exists n\,(m + n = 0)$

$(c)\ \ \forall m\ \forall n\ \forall p\,\big(m + (n + p) = ((m + n) + p)\big)$

$(d)\ \ \forall m\ \exists n\,(m < n)$

$(e)\ \ \forall m\ \forall n\,\big(((m \geq 0) \wedge (n \geq 0)) \rightarrow (mn \geq 0)\big)$

$(f)\ \ \forall m\ \forall n\ \exists p\,(mn = p)$

Solution:

(a) $m + n = n + m$ For all real numbers m *and* n

This is the commutative law for addition of real numbers. Or for every real numbers m *and* n, $m + n = n + m$

(b) For every real numbers m, there is a real number n such that $m + n = 0$. this states that every real number has an additive inverse.

(c) This is the associative law for addition of real numbers.

i.e., for every real numbers m, n *and* p,

$$m + (n + p) = (m + n) + p$$

(d) For every real numbers m, there exists is a real number n such that m is less than n.

(e) For every real numbers m and real number n, if m and n are both non-negatives, then their product is non-negative.

(f) For every real numbers m and real number n, there exists is a real number p such that $mn = p$.

Example 2.26

Translate into English the statement $6\,m\;6\,n\left(\left(m>0\right)\hat{}\left(n<0\right)\rightarrow\left(mn<0\right)\right)$,

Where the universe of discourse for both variables consists of all real numbers.

Solution:

This statement says that for every real numbers m *and* n, if m is positive and n is negative, then mn is negative. In other words, "The product of a positive real number and a negative and a negative real number is a negative real number".

Example: 2.27.

Translate the statement $6m\left(C\left(m\right)\text{ v }\mathrm{E}n\left(C\left(n\right)\hat{}\,F\left(m,n\right)\right)\right)$

Into English, where $C\left(m\right)$ is "m has a computer". $F\left(m,n\right)$ is "m *and* n are friends", and the universe of discourse for both m *and* n consists of all students in your school.

Solution:

The above statement says that for every student m in your school m has a computer or there is a student n such that n has a computer and m *and* n are friends.

In other words, every student in your school has a computer or has a friend who has a computer .

Example : 2.28.

Let $P\left(m,n\right)$ be the statement "m has sent an e – mail message to n", where the universe of discourse for both m *and* n consists of all students in your class. Express each of these quantifications in English.

$\left(a\right)\mathrm{E}\,m\;\mathrm{E}\,n\;P\left(m,n\right)$

$\left(a\right)6\;m\;\mathrm{E}\,n\;P\left(m,n\right)$

$\left(a\right)6\;m\;6\;n\;P\left(m,n\right)$

Solution:

[a] There is some student in your class who has sent a message to some student in your class.

[b] Every student in your class has sent a message to atleast one student in your class.

[c]Every student in the class has sent a message to every student in the class.

Example : 2.29

Translate the statement $\mathrm{E}\;m\;6\,n\,p\left(F\left(m,n\right)\hat{}\,F\left(m,p\right)\hat{}\left(n\text{ s }p\right)\rightarrow\neg F\left(n,p\right)\right)$

Into English, where $F\left(m,n\right)$ means that m *and* n are friends and the universe of discourse for m , n and p consists of all student in your class.

Solution :

First examine the expression

$$\left(F\left(m,n\right)\hat{}\,F\left(m,p\right)\hat{}\left(n\text{ s }p\right)\rightarrow\neg F\left(n,p\right)\right)$$

i.e., If students m *and* n are friends, and students m and p are friends and if n *and* p are not the same student, then n *and* p are not friends.

Thus the original student becomes, there is a student m such that for all students n and all students p ,if m *and* n are friends and m and p are friends, then n *and* p are not friends.

In other words, there is a student none of whose friends are also friends with each other.

2.3.2 Translating Sentences into Logical Expressions

This section shows that how quantifiers can be used to translate sentences into logical expressions.

Example 2.30.

Express the statement "If a person is male and is a parent, then this person is someone's father" as a logical expression involving predicates, quantifiers with a universe of discourse consisting of all people , and logical connectives.

Solution:

The given statement can be rewritten as

"For every person p , if person p is male and person p is a parent, then there is a person q such that person p is the father of person q ".

Let us introduce the predicate as follows:

Let $M(p)$ to represent " p is male",

$P(p)$ to represent " p is a parent" and $F(p,q)$ to represent " p is the father of q ".

Hence the original statement can be represented as

$$6\,p\left(\left(M(p) \wedge P(p)\right) \to \mathrm{E}qF(p,q)\right)$$

Since q does not appear in $M(p) \wedge P(p)$, the above expression becomes

$$6\,q\,\mathrm{E}\,q\left(\left(M(p) \wedge P(p)\right) \to F(p,q)\right)$$

Example : 2.31.

Express the statement "Everyone has exactly one best friend" as a logical expression involving predicates, quantifiers with a universe of discourse consisting of all people, and logical connectives.

Solution :

The above statement can be expressed as "For every person p has exactly one best friend".

Introducing the universal quantifier, the above statement can be rewritten as

"$6p\,(Person\ p\ has\ exactly\ one\ best\ friend)$", Where the universe of discourse consist of all people. Now, "$p\ has\ exactly\ one\ best\ friend$" means that "there is a person q who is the best friend of p and that for every person r , if person r is not person q , then r is not the best friend of p ".

Let us introduce the predicate $F(p,q)$ be the statement " q is the best friend of p ".

The statement that " p has exactly one best friend" can be represented as

$$\exists q\left(F\left(p,q\right)\wedge\forall r\left(\left(r\neq q\right)\rightarrow\neg F\left(p,r\right)\right)\right)$$

Thus, the given statement can be expressed as $\forall p\,\exists q\left(F\left(p,q\right)\wedge\forall r\left(\left(r\neq q\right)\rightarrow\neg F\left(p,r\right)\right)\right)$

Example: 2.32

Use quantifiers to express the statement "there is a man who has taken a flight on every airline in the world."

Solution :

Let $A(m,f)$ be"$m has taken\ f$" and $B\left(f,a\right)$ be"$f\ is\ a\ flight\ on\ a$".

We can express the given statement as

$$\exists m\,\forall a\,\exists f\left(A(m,f)\wedge B\left(f,a\right)\right),$$

Where the universe of discourse for $w, f\ and\ a$ consist of all the men in the world, all airplane flights and all airlines respectively.

Thus, the given statement can be expressed as $\exists m\,\forall a\,\exists f\,C\left(m,a,f\right)$, where $C\left(m,a,f\right)$ is "$m\ has\ taken\ f\ on\ a$". but the first solution is usually preferable, since the second solution is more compact.

Example: 2.33

Express the following statement into a logical expression form "the sum of two negative integers is negative".

Solution :

The given statement can also be expressed in the following form.

"For every two negative integers, the sum of these integers is negative".

i.e. "For all negative integers $m\ and\ n$, $m+n$ is negative."

Hence we can express the given statement as

$$\forall m\,\forall n\left(\left(m<0\right)\wedge\left(n<0\right)\rightarrow\left(m+n<0\right)\right)$$

Where the universe of discourse for both variables consists of all integers.

Example: 2.34

Consider the statement "every real number except zero has a multiplicative inverse". Translate it into a logical expression.

Solution :

We can write the given statement as

"For every real number r except zero, r has a multiplicative inverse".

i.e. "For every real number r, $if\ r\neq 0$, then there exists a real number s such that $rs=1$."

This can be rewritten as

$$\forall r\left(\left(r\neq 0\right)\rightarrow\exists s\left(rs=1\right)\right)$$

Where the universe of discourse for both variables consists of all integers.

Negative Nested Quantifiers

By applying the rules for negating statements involving a single quantifier, we can negate the statements involving nested quantifier.

Example 2.35

Express the negation of the statement $\forall m\ \exists n\ (mn = 1)$

Solution :

$$\neg\forall m\ \exists n\ (mn = 1)$$
$$\equiv \exists m\ \neg\exists n\ (mn = 1)$$
$$\equiv \exists m\ \forall n\ \neg(mn = 1)$$
$$\equiv \exists m\ \forall n\ (mn \neq 1)$$

Which is the negation of the given statement.

Example 2.36

Express the negation of each of these statements so that all negation symbols immediately precede predicates.

$(a)\ \forall m\ \exists n\ \forall r\ A(m,n,r)$

$(b)\ \forall m\ \exists n\ B(m,n) \vee \forall m\ \exists n\ C(m,n)$

$(c)\ \forall m\ \exists n\ (B(m,n) \wedge \exists r\ D(m,n,r))$

$(d)\ \forall m\ \exists n\ (B(m,n) \rightarrow C(m,n))$

Solution

$(a)\quad \neg\forall m\ \exists n\ \forall r\ A(m,n,r)$

$\qquad \equiv \exists m\ \forall n\ \exists r\ \neg A(m,n,r)$

Which is the required expression

$(b)\quad \neg\left[\forall m\ \exists n\ B(m,n) \vee \forall m\ \exists n\ C(m,n)\right]$

$\qquad \equiv \neg\left[\forall m\ \exists n\ B(m,n)\right] \wedge \neg\left[\forall m\ \exists n\ C(m,n)\right]$

$\qquad \equiv \exists m\ \forall n\ \neg B(m,n) \wedge \exists m\ \forall n\ \neg C(m,n)$

Which is the required expression

$(c)\quad \neg\forall m\ \exists n\ (B(m,n) \wedge \exists r\ D(m,n,r))$

$\qquad \equiv \exists m\ \forall n\ \neg(B(m,n) \wedge \exists r\ D(m,n,r))$

$\qquad \equiv \exists m\ \forall n\ (\neg B(m,n) \vee \forall r\ \neg D(m,n,r))$

Which is the required expression

$$(d) \quad \neg \forall m \, \exists n \left(B(m,n) \to C(m,n)\right)$$

$$\equiv \exists m \, \forall n \neg B(m,n) \to C(m,n)$$

$$\equiv \exists m \, \forall n \neg \left(\neg B(m,n) \wedge C(m,n)\right)$$

$$\equiv \exists m \, \forall n \left(B(m,n) \vee \neg C(m,n)\right)$$

Which is the required expression

Example 2.37

Use quantifiers to express the statement that "There does not exist a man who has taken a flight on every airline in the world".

Solution :

By example [2.32] , our statement can be expressed as

$$\neg \exists m \, \forall a \, \exists f \, \left(A(m,f) \wedge B(f,a)\right),$$

Where $A(m,f)$ is "m *has taken* f" and $B(f,a)$ is "f is a fight on a"

Successively applying the rules for negated quantified statements, we get

$$\neg \exists m \forall a \exists f \left(A(m,f) \wedge B(f,a)\right),$$

$$\equiv \forall m \, \exists a \forall f \neg A(m,f) \wedge B(f,a)$$

$$\equiv \forall m \, \exists a \forall f \left(\neg A(m,f) \vee \neg B(f,a)\right)$$

Hence the result.

2.3.4 The order of Quantifiers

Most of the mathematical statements consist of multiple quantifications of propositional functions involving more than one variable. The order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers. In the following examples the universe of discourse for each variable consists of all real numbers.

Example 2.38

Let $A(m,n)$ be the statement "$m + n = n + m$". What is the truth value of the quantification

$\forall m \, \forall n \, A(m,n)$?

Solution:

$A(m,n)$ is true for real numbers m *and* n.

Hence the proposition $\forall m \, \forall n \, A(m,n)$ *is true.*

Example 2.39

Let $B(m,n)$ denote "$m + n = 0$". What are the truth values of the quantifications

$\exists n \forall m \; B(m,n)$ *and* $\forall m \exists n B(m,n)$.

Solution: The quantification $\exists n \forall m \; B(m,n)$ denotes the proposition

"There is a real number n such that for every real number $m, m+n=0$"

There is only one value of m for which $m+n=0$. Since there is no real number n such that $m+n=0$ for all real numbers m, the statement $\mathrm{E}n\,6\ m\quad B(m,n)$ is false.

The quantification $6\ m\ \mathrm{E}\ n\,B(m,n)$ denotes the preposition.

"For every real number m there is a real number n such that $m+n=0$"

Given a real number m, there is a real number n such that $n=-m$. Hence, the statement $6\ m\ \mathrm{E}\ n\,B(m,n)$ is true.

Example 2.40

Let $R(m,n,r)$ be the statement "$m+n=6$". What are the truth values of the statements

$$6m\ 6n\ L\ rR(m,n,r)\ \ and\ \ \mathrm{E}\ r\ 6m\ 6n\ R(m,n,r)$$

Solution :

The quantification $6m\ 6n\ \mathrm{E}\ rR(m,n,r)$ is the statement

"For all real numbers m and for all real numbers n there is real number r such that "$m+n=r$" is true.

The quantification $\mathrm{E}\ r\ 6m\ 6n\ R(m,n,r)$ is the statement.

"There is real number r such that for all real numbers m and for all real numbers n it is true that $m+n=r$".

Since there is no value of r that satisfies the equation $m+n=r$ for all values of m and n, the quantification $\mathrm{E}\ r\ 6m\ 6n\ R(m,n,r)$ is false.

2.4 PROOF TECHNIQUES

2.4.1 Introduction to Proof

In the preceding sections, we discussed the various ways of using logical arguments and deriving conclusions. In Mathematics and Computer Science, mathematical logic is used to prove theorems and correctness of program. In this section, after formally defining the term "theorem", we describe some techniques that are used in proving theorems.

A theorem is a statement that can be shown to be true [under certain conditions]. For example, consider the following statement:

If x is an integer and x is odd , then x^2 is odd.

Or

For all integer x ,if x is odd , then x^2 is odd.

This statement can be shown to be true.

Theorems are typically defined as follows:

As facts. For example, 5 is an odd integer . As another example , the equation $x^2+2=0$ has no solutions in real numbers.

As implications. For example, for all integers x ,if x is odd , then $x+1$is even.

As biimplications. For example, for all integers x ,if x is odd if and only if x is not divisible by 2.

A proof may consist of previously known facts, proved results, or previous statements of the proof. To construct proofs, methods are needed to derive new statements from odd ones. The statements used in a proof can include axioms or postulates. This section illustrates few techniques for constructing a proof.

The terms lemma and corollary are used for certain types of theorems. A lemma is a simple theorem used in the proof of other theorems. A corollary is propositions that can be established directly form a theorem. A conjecture is a proposition whose truth value is unknown. If a proof of a conjecture is found, then the conjecture becomes a theorem.

The proof techniques used in this section are important not only to prove mathematical theorems, but also for their many applications to computer science viz. verifying that the computer programs are correct or not, establishing that operating systems are secure, making inferences in the area of artificial intelligence, showing that system specifications are correct, and so on.

2.4.2 Proof Methods

We now introduce different proof methods. The techniques for proving implications are important, because many theorems are in implications form. We know that $p \rightarrow q$ is true unless p is true but q is false . In order to prove the statement $p \rightarrow q$, it is enough to prove that q is true if p is true. The following discussions will give the most common techniques for proving theorems.

[1] Direct proof

We first discuss the proof of those theorems that can be expressed in the form.

$$6x\left(P(x) \rightarrow Q(x)\right), D \text{ is the domain of discourse} \qquad [2.5]$$

For example , for all integers x ,if x is odd, then $x+1$ is even. To construct a proof of the theorem [2.5], first we select an arbitrary member 't' of the domain D. Then we show that the statement $P(t) \rightarrow Q(t)$ is true. For this we assume that $P(t)$ is true. Then show that $Q(t)$ is true. If we do this, then by the rule of universal generalization, $(6x)\left(P(x) \rightarrow Q(x)\right), D$ is the domain of discourse is true. This procedure is called the proof by direct method or direct method.

In other words, the implication $P \rightarrow Q$ can be proved by showing if p is true, then Q must also be true. A proof of this kind is called a direct proof.

Definition 2.1

The integer m is even if there exists an integer k such that $m = 2k$, and it is odd if there exists an integer k such that $m = 2k+1$

Example 2.41

Give a direct proof of the theorem "For all integers m ,is odd then m^2 is odd".

Solution :

Let $P(m)$ denote "m "is an odd integer" and $Q(m)$ denote "m^2 is an odd integer".

Then the symbolic form of the above theorem is as follows :

$6\,x\,m\left(P\left(m\right)\rightarrow Q\left(m\right)\right)$, the domain of discourse is the set Z of all integers.

Assume that 'a'is an arbitrary chosen element of Z. Also , assume $P\left(a\right)$ is true.

To prove that $Q\left(a\right)$ is true, Let 'a' be an integer such that a is odd.

Then we can write $a = 2k+1$ for some integer k.

This implies that $a^2 = \left(2k+1\right)^2 = 4k^2+4k+1 = 2\left(2k^2+2k\right)+1 = 2m+1$, where $m = 2k^2+2k$ is an integer.

Therefore a^2 is an integer.

Thus, for all integers x, if x is odd, then x^2 is odd.

Note: 2.4

Verifying a statement for a particular value is not a proof. It only indicates that the theorem might be true. If we are not careful, then verifying a statement for a particular value of the variable can give a weong indication. For example, consider the following sentence:

"For all real numbers $k, k^2 \geq k$"

If we choose k to be an integer greater than or equal to 1 or less than or equal to -1 or k to be zero, then the sentence is true. If we choose k to be $\frac{1}{2}$, $then\ k^2 = \frac{1}{4} < \frac{1}{2} = k$

We must be careful when writing proofs.

Example 2.42

Give a direct proof of the theorem "For all integers m and n, If m and n are odd, then the product mn is odd".

Solution :

Let us change it into symbolic form. Let $P\left(m\right)$ denote "m is an odd integer", $Q\left(n\right)$ denote "n is an odd integer" and $R\left(m,n\right)$ denote "the product mn is an odd integer".

Then the above theorem becomes $6\ m\ 6\ n\left(P\left(m\right)\hat{}Q\left(n\right)\right)\rightarrow R\left(m,n\right)$, the domain of the discourse is the set Z of all integers.

Suppose that $a\ and\ b$ are odd integers.

Then $a = 2k+1\ and\ b = 2l+1$ for some integers $k\ and\ l$.

Now

$$ab = \left(2k+1\right)\left(2l+1\right)$$

$$, \quad = 4kl+2k+2l+1$$

$$= \left(2kl+k+l\right)+1$$

$= 2t+1, where\ t = 2kl+k+l$ is an integer.

$\therefore ab$ is odd.

Therefore, for all integer m and n ,if m and n are odd, then the product mn is odd.

[2] Indirect Proof

Consider the implication $p \to q$ .In order to show that $p \to q$ is true , it is enough to show that the implication $\neg p \to \neg q$ is true, since the implication $p \to q$ is equivalent to the implication $\neg p \to \neg q$. Now to show that $\neg p \to \neg q$ is true , we assume that $\neg q$ is true and prove that $\neg p$ is true. This type of proof is called an indirect proof.

Example 2.43

Give an indirect proof of the following theorem."Let m be an integer. If $m^2 + 3$ is odd, then m is even".

Solution:

Let $P(m)$ denote "$m^2 + 3$ is an odd integer"

and $Q(m)$ "m is an even integer".

Thus the theorem becomes

$\forall m\,(P(m) \to Q(m))$, the domain of discourse is the set Z of all integers.

Assume that m is a particular but arbitrarily chosen element of Z .

For this m ,we have to show that $P(m) \to Q(m)$ is true, i.e, it is enough to show that $\neg Q(m) \to \neg P(m)$ is true.

Suppose that $\neg Q(m)$ is true.

$\therefore$ *m is not even.*

$\therefore$ *m is odd.*

So, $m = 2k + 1$ *for some integer* k.

$$
\begin{aligned}
Now,\ m^2 + 3 &= (2k+1)^2 + 3 \\
&= 4k^2 + 4k + 1 + 3 \\
&= 4k^2 + 4k + 4 \\
&= 2(2k^2 + 2k + 2)
\end{aligned}
$$

$= 2n$, *when* $n = 2k^2 + 2k + 2$ and n is integer

$m^2 + 3 = 2n$ for some integer n

$\therefore$ $m^2 + 3$ is an even integer i.e.,

$m^2 + 3$ is not an odd integer.

Thus $\neg P(m)$ is true.

Hence, $\neg Q(m) \to \neg P(m)$ is true.

$\forall m\,(P(m) \to Q(m))$,in the domain Z of all integers, by the rule of universal generalization.

i.e., if $m^2 + 3$ is an odd integer, then m is even.

Example 2.44

Give an indirect proof of the theorem, "If $5m + 2$ is odd, then m is odd".

Solution :

Given statement is written is symbolic form as

$6\ m\ P(m) \to Q(m)$,the domain of discourse is the set Z of all integers.

Where $P(m)$ denotes "$5m+2$ is an odd integer"

and $Q(m)$ denotes "m is an even integer"

Let m be any arbitrary value of Z

To prove that $P(m) \to Q(m)$ is true.

It is enough to prove that $7Q(m) \to 7P(m)$ is true.

Assume that $7Q(m)$ is true.

i.e., assume that m is even

Then $m = 2n$ for some integer n .

To prove $5m+2$ is an even integer

Now,

$$\begin{aligned} 5m+2 &= 5(2n)+2 \\ &= 10n+2 \\ &= 2(5n+1) \\ &= 2t, \textit{ where } t = 5n+1 \text{ is an integer.} \end{aligned}$$

Thus, $5m+2$ is an even integer

$5m+2$ is not odd .

Thus, $7Q(m) \to 7P(m)$ is true.

Hence, by the rule of universal generalization,

$6\ m\ P(m) \to Q(m)$ is true.

If $5m+2$ is not odd ,then m is odd.

[3] Vacuous Proof

Assume that the hypothesis P of an implication $P \to Q$ is false. Then the implication $P \to Q$ is true, since the statement has the form $F \to T$ or $F \to F$,which is true. Hence, if it can be shown that P is false, then a proof of $P \to Q$ can be given. This proof is called a vacuous proof. Vacuous proof are used to establish a theorem of the kind $6\ m\ P(x)$,where $P(x)$ is the propositional function.

Example 2.45

Let $P(m)$ be the proportional function "if $m > 1$, then $m^2 > m$". Show that the proposition $P(0)$ is true.

Solution :

$P(0)$ is the proposition "If $0 > 1$, then $0^2 > 0$". But, the hypothesis $0 > 1$ is false.

Hence the implication $P(0)$ is true.

Note. 2.5

The conclusion of the above implication is false, which is irrelevant to the truth value of the implication.

[4] Trivial Proof

Assume that the conclusion Q of an implication $P \rightarrow Q$ is true. Then the implication $P \rightarrow Q$ is true, since the statement has the form $T \rightarrow T$ or $F \rightarrow T$, which is true. Hence, if it can be shown that Q is true, then a proof of $P \rightarrow Q$ can be given. This proof is called a trivial proof. Trivial proofs are important in Mathematical induction and when special cases of theorems are proved.

Example 2.46

Let $P(m)$ be the propositional function "if $a \text{ and } b$ are positive integers with $a \geq b$,then $a^m \geq b^m$. Show that the proposition $P(0)$ is true.

Solution

The proposition $P(0)$ is "if $a \text{ and } b$ are positive integers with $a \geq b$, then $a^0 \geq b^0$". Since $a^0 = 1$ and $b^0 = 1$, the conclusion of $P(0)$ is true.

Hence the proposition $P(0)$ is true.

[5] A little proof strategy

Suppose that the theorem is asked to prove, which method should you use? First, quickly evaluate whether a direct proof is applicable. If not, try the same thing with an indirect proof. Sometimes when there is no obvious way to approach a direct proof, an indirect proof, an indirect proof will work. We illustrate this strategy in the following examples.

Definition 2.2

The real number r is rational if there exist integers $a \text{ and } b$ with $b \neq 0$ such that $r = \frac{a}{b}$. A real number which is not rational is called irrational.

Example 2.47

Prove the sum of two rational numbers is rational.

Solution :

First, we try for a direct proof.

Let $p \text{ and } q$ be two rational numbers.

From the definition of rational numbers, there exists integers $a \text{ and } b$ with $b \neq 0$ such that

$p = \frac{a}{b}$ and there exists integer $c \text{ and } d$ with $d \neq 0$ such that $p = \frac{c}{d}$.

Then $p + q = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

Since $b \neq 0$, $d \neq 0$,it follows that $bd \neq 0$ .Also, we expressed $p + q$ as the ratio of two integers $ad + bc \text{ and } bd$, with $bd \neq 0$.

Hence, $p+q$ is rational.

Thus, our direct proof is succeeded.

Example 2.48

Prove that if an m is an integer and m^2 is odd, then m is odd.

Solution :

We first try for a direct proof.

Assume that m is an integer and m^2 is odd. Then , there exists an integer p such that

$m^2 = 2p+1 \Rightarrow m = \pm\sqrt{2p+1}$

Can we use this information to show that m is odd? No, this is not at all useful.

Hence, we try for the indirect proof.

Assume that the integer m is not odd.

m is even.

$\therefore$ There exists an integer p such that $m = 2p$.

To prove that m^2 is not odd.

i.e., to prove that m^2 is even.

Now, $m^2 = \left(2p\right)^2 = 4p^2 = 2\left(2p\right)^2 = 2t,\ where\ t = 2p^2 \Rightarrow m^2$ is even.

Thus, "if m^2 is an odd integer then m is odd". Hence, an indirect proof is succeeded.

[6] Proof by contradiction

In this method, we assume that the conclusion is not true and we arrive at a contradiction.

Example 2.49 Using proof by contradiction, show that $\sqrt{2}$ is not an irrational number.

Then $\sqrt{2}$ is rational.

We have to show that this leads to a contradiction. Since $\sqrt{2}$ is rational, there exists integers $p\ and\ q$ with $q \neq 0$ such that $\sqrt{2} = \frac{p}{q}$. We assume that $\frac{p}{q}$ is in the lowest term. i.e., $p\ and\ q$ have no common factors other than 1.

Now, $\sqrt{2} = \frac{p}{q}$.

Squaring on both sides, Wet get

$$\left(\sqrt{2}\right)^2 = \left(\frac{p}{q}\right)^2$$

$$\Rightarrow \quad 2 = \frac{p^2}{q^2}$$

$\Rightarrow p^2 = 2q^2$

$\Rightarrow p^2$ is an even integer.

$\therefore$ p is an even integer, since we know that if p is odd then p^2 is also odd.

$p = 2k$, for some integer k.

$Thus, (2k)^2 = 2q^2$

$i.e., \quad 4k^2 = 2q^2$

$\therefore \quad q^2 = 2k^2$

This means that q^2 is even.

Hence, q must be even

Thus, we have proved that both p *and* q are even.

They have 2 as a common factor.

This contradicts our assumption that p *and* q have no common factor other than 1.

Thus we arrived at a contradiction.

Hence, $\sqrt{2}$ is an irrational number.

Example : 2.50

Give a proof by contradiction of the theorem "Let A *and* B be sets such that $A \subset B$ *Then* $A \cap B = A$".

Solution

Let A *and* B be any two sets such that $A \subset B$.To prove that $A \cap B = A$

Assume that $A \cap B \neq A$

Then $A \cap B \subset A$

If $A \cap B \neq A$, then $A \not\subset A \cap B$

$\therefore$ there exists an element $a \in A$ such that $a \notin A \cap B$

Now , $a \notin A \cap B$

$\therefore$ $a \notin A$ or $a \notin B$

Hence, $a \in A$ and $a \notin A \cap B$

$\therefore$ $a \notin B$

We found an element a such that $a \in A$ but $a \notin B$

$\therefore$ $A \not\subset B$

This is a contraction to our hypothesis $A \subset B$ .Our assumption is wrong. Hence, if $A \subset B$, then $A \cap B = A$.

Example 2.51

Show that at least four of any 22 days must fall on the same day of the week.

Solution :

Let P be the proposition "At least four of the 22 chosen days are the same day of the week".

Suppose that P in true.

Then at most three of the 22 days are the same day of the week.

Since there are seven days in a week , at most 21 days could have been chosen. We have one more day in a week. This is a contradiction to our assumption.

Our assumption is wrong. Hence the result.

Example 2.52

Give a proof by contradiction of the theorem if $3m+2$ is odd, then m is odd".

Solution :

Assume that $3m+2$ is odd and m is not odd. Thus m is even.

$m = 2k,$ for some integer ' k '.

Now,

$$\begin{aligned} 3m+2 &= 3(2k)+2 \\ &= 6k+2 \\ &= 2(3k+1) = 2t, \text{ where } t = 3k+1 \text{ is an integer} \\ 3m+2 &\text{ is even} \end{aligned}$$

This is a contradiction to our assumption that $3m+2$ is odd.

? Our assumption is wrong.

Hence, "if $3m+2$ is odd, then m is odd".

[7] Proof by cases

Consider the implication of the form

$$(P_1 \vee P_2 \vee \ldots \vee P_m) \to Q$$

To prove this the tautology

$$[(P_1 \vee P_2 \vee \ldots\ldots \vee P_m) \to Q] \leftrightarrow [(P_1 \to Q) \wedge (P_2 \to Q) \wedge \ldots\ldots \wedge (P_m \to Q)]$$

Can be used as a rule of inference.

That is , if the hypothesis of the original implication made up of m propositions P_1 , P_2 ,........P_m can be proved by proving each of the m implications $P_1 \to Q, P_2 \to Q, \ldots\ldots\ldots P_m \to Q$ individually. Such an argument is called a proof by cases. In certain cases to prove an implication $P \to Q$ is true, it is convenient to use $P_1 \vee P_2 \vee \ldots\ldots \vee P_m$ instead of P ,Where P and $P_1 \vee P_2 \vee \ldots\ldots \vee P_m$ are equivalent.

Example 2.53

Using a proof by cases, show that $|r_1 r_2| = |r_1||r_2|$,where r_1 *and* r_2 are real numbers.

Solution:

We know that

$$|r| = \begin{cases} -r, & \text{when } r < 0 \\ r, & \text{when } r \geq 0 \end{cases}$$

Let the proposition P be " r_1 *and* r_2 are real number" and let Q be " $|r_1 r_2| = |r_1||r_2|$ ".

Let

P_1 be "$r_1 \geq 0 \wedge r_2 \geq 0$",

P_2 be "$r_1 \geq 0 \wedge r_2 < 0$",

P_3 be "$r_1 < 0 \wedge r_2 \geq 0$",

P_4 be "$r_1 < 0 \wedge r_2 < 0$",

Hence, P is equivalent to $\left(P_1 \vee P_2 \vee P_3 \vee P_4\right)$

To show that $P \to Q$.

i.e., it is enough to prove that $P_1 \to Q, P_2 \to Q, P_3 \to Q,$ and $P_4 \to Q$

(i) $when$ $r_1 \geq 0$ and $r_2 \geq 0$; $\quad r_1 r_2 \geq 0$

$\therefore |r_1 \ r_2| = r_1 \ r_2 = |r_1| \ |r_2|$

$Thus,$ $P_1 \to Q$

(ii) $when$ $r_1 \geq 0$ and $r_2 < 0$; $\quad r_1 r_2 \leq 0$

$\therefore |r_1 \ r_2| = -\left(r_1 \ r_2\right) = r_1\left(-r_2\right) = |r_1| \ |r_2|$

$Thus,$ $P_2 \to Q$

(iii) $when$ $r_1 < 0$ and $r_2 \geq 0$; $\quad r_1 r_2 \leq 0$

$\therefore |r_1 \ r_2| = -\left(r_1 \ r_2\right) = \left(-r_1\right) r_2 = |r_1| \ |r_2|$

$Thus,$ $P_3 \to Q$

(iv) $when$ $r_1 < 0$ and $r_2 < 0$; $\quad r_1 r_2 > 0$

$\therefore |r_1 \ r_2| = r_1 \ r_2 = \left(-r_1\right)\left(-r_2\right) = |r_1| \ |r_2|$

$Thus,$ $P_4 \to Q$

Hence the theorem.

[8] Proving bi implications

Some theorems can be expressed by using universal quantifiers and logical connective biimplication. We already proved that the bi implication $P \leftrightarrow Q$ is equivalent to $\left(P \to Q\right) \wedge \left(Q \to P\right)$. Hence to prove $P \leftrightarrow Q$, it is enough to prove that the implications $P \to Q$ and $Q \to P$ are true.

Example 2.54

Prove that "an integer m is even if and only if $m+1$ is odd".

Solution :

To prove this , we have to do the following : first, assume that m is even and show that $m+1$ is odd. Then assume that $m+1$ is odd and prove that m is even.

Let us assume that m is even . Then $m = 2k$, for some integer k .

$m+1 = 2k+1 \Rightarrow m+1$ is an odd integer.

Now, assume that $m+1$ is odd. Then $m+1 = 2k+1$,for some integer k .

$m = 2k \Rightarrow m$ is an even integer.

Hence, an integer m is even if and only if $m+1$ is odd.

Example 2.55

Prove the following theorem

Let A *and* B be non – empty subsets of a set X . Then $A = B$ if and only if $A \times B = B \times A$.

Solution :

To prove the above theorem, first we show that if $A = B$,then $A \times B = B \times A$ .Then we show that if, $A \times B = B \times A$, then $A = B$

Suppose that $A = B$.

Then

$$A \times B = A \times A \ and \ B \times A = A \times A$$
$$\Rightarrow A \times B = B \times A$$

Now suppose that $A \times B = B \times A$

To show that $A = B$

i.e., it is enough to prove that every element of A is an element of B and every element of B is element of A .

i.e., to prove that $A \subseteq B$ and $B \subseteq A$.

$Let \ (x \in A)$.Since $B \neq \emptyset$, there exists an element $y \in B$

$$Then \quad (x, y) \in A \times B$$
$$\Rightarrow \quad (x, y) \in B \times A, \text{since } A \times B = B \times A$$
$$\Rightarrow \quad (x, y) = (u, v), \ for \ some \ u \in B \ and \ v \in A,$$
$$\Rightarrow \quad x = u \ and \ y = v$$
$$Thus, \quad x = u \in B$$

Since x is an arbitraty element of $A, A \subseteq B$

Now , we have to show that $B \subseteq A$

Let $b \in B$. Because $A \neq \emptyset$, there exists an element $a \in A. \ Then \ (a, b) \in B \times A \Rightarrow (b, a) \in A \times B$ since

$$B \times A = A \times B$$
$$(b, a) \in A \times B$$
$$\Rightarrow \quad (b, a) = (m, n) \ for \ some \ m \in A \ and \ n \in B.$$
$$\Rightarrow \quad b = m \ and \ a = n$$
$$Thus, \ b = m \in A$$

Since x is an arbitraty element of $B, B \subseteq A$

Hence , from equations [2.6] and [2.7],

$$A = B$$

[9] Proving equivalent statements:

Sometimes theorem are in the form of equivalent statements. How do we prove this type of theorem For example, consider the theorem that says that the statements P , Q ,and R are equivalent. For this normally we show that $P \to Q, Q \to R, \ and \ R \to P$. That is , we have a cycle $P \to Q \to R \to P$. This is not the only way.

We can also prove that P if and only if Q ,and then Q if and only if R . There are also other ways prove a theorem that consists of equivalent statements. We can prove that P if and only if R , and Q and only if R ,[or] P if and only if Q and P if and only if R .

Example 2. 56

Let k be an integer . Then prove that the following statements are equivalent : $P : k$ is divisible by 6

$Q : k$ is divisible by 2 and 3

$R : k$ is an even number and is divisible by 3.

Solution

To prove that these three statements are equivalent.

i.e., it is enough to prove that the implications $P \to Q, Q \to R,$ *and* $P \to R$ are true.

[i] $P \to Q$

Suppose that k is divisible by 6.Then $k = 6m$ for some integer m .

Then $k = 6m = 2(3m) = 3(2m)$

$\therefore$ k is divisible by 2 and 3.

$P \to Q$

[ii] $Q \to R$

Suppose that k is divisible by 2 and 3.Since k is divisible by 2, k is an even integer. Hence, k is an even integer and is divisible by 3.

$Q \to R$

[iii] $Q \to R$

Suppose that k is an even number and is divisible by 3.

Since k is an even number, $k = 2m$ for some integer m .

$\therefore$ $(2m)$ is divisible by 3.

$2m = 3n$,for some integer n .

Now,

$$m = 3m - 2m$$
$$= 3m - 3n$$
$$i.e., m = 3(m-n)$$
$$= 3t, where\ t = m - n$$

Is an integer

i.e., $m = 3t =$, for some integer t

$k = 2m = 2(3t) = 6t$ for some integer t .

$\therefore$ k is divisible by 6

$R \to P$

Hence the proof.

Example 2. 57

Show that the following statements are equivalents

P : m is an even integer

Q : m - 1 is an odd integer

R : m^2 is on even integer.

Solution

In order to prove the above result, it is enough to prove that the implications $P \rightarrow Q, Q \rightarrow R,$ *and* $R \rightarrow P$ are true.

To prove $P \rightarrow Q$ is true

Suppose that m is an even integer. Then $m = 2u$, for some integer u .

Now, $m = -1 = 2u - 1 = 2\left(u-1\right) + 1 = 2t + 1,$

where $t = u - 1$ is an odd integer.

This means that $m - 1$ is an odd integer.

$P \rightarrow Q$

To prove $Q \rightarrow R$ is true

Suppose that $m - 1$ is an odd integer.

$Then \quad m - 1 = 2u + 1 \; for \; some \, \text{int}\, eger \, u.$

⍰ $\qquad m = 2u + 2$

$$m^2 = \left(2u+2\right)^2 = 4u^2 + 8u + 4 = 2\left(2u^2 + 4u + 2\right)$$

$$= 2s, where \;\; s = 2u^2 + 4u + 2 \; is\, an \;\text{int}\, eger.$$

⍰ $\qquad m^2 \;\; is\, even.$

Hence, $Q \rightarrow R$

To prove $P \rightarrow R$ is true

Let us use indirect proof.

Assume that m is not even.

To prove that m^2 is not even.

Since m is not even, m must be odd.

⍰ m^2 is odd, since we know that “if m is odd, then m^2 is odd”.

Hence $P \rightarrow R$.

Hence the result.

2.4.3 Fallacies [Errors] in the Proofs

As stated earlier, a proof may consist of previously known facts, proved results or previous statements of the proof . However, if we are not careful, errors can occurs in the proofs. We will briefly describe some of these here. Most common errors are mistakes in arithmetic, basic algebra and when working with complicated formulae.

Example 2.58

Find the error in the following "Proof" that $1 = -1$.

$$\Pr oof : 1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{(-1)}.\sqrt{(-1)} = \left[\sqrt{(-1)}\right]^2 = -1$$

Solution

$1 = -1$ is not true.

So, where is the error?

The error is in the inequality $\sqrt{(-1).(-1)} = \sqrt{(-1)}.\sqrt{(-1)}$

Here, we used the fact that $\sqrt{xy} = \sqrt{x}.\sqrt{y}$ for all real numbers x *and* y. However, the expression $\sqrt{xy} = \sqrt{x}.\sqrt{y}$ is true when x *and* y are non – negative real numbers.

Hence, we must be very careful while writing the proofs.

Example 2. 59

Find the error in the following "Proof" that $1 = 2$.

Proof :

Let m and n be two equal positive integers. We use the following steps.

Step	Re*ason*
$(1)\ m = n$	*Given*
$(2)\ m^2 = mn$	*Multiply both sides of* (1) *by* m
$(3)\ m^2 - n^2 = mn - n^2$	*Subtract* n^2 *formboth sides of* (2)
$(4)\ (m-n)(m+n) = n(m-n)$	*Factor both sides of* (3)
$(5)\ m + n = n$	*Divideboth sides of* (4) *by* $m - n$
$(6)\ 2n = n$	Re*place m by n in* (5) *and simplify*
$(7)\ 2 = 1$	*Divideboth sides of* (6) *by n.*

Solution

In step [5] , we divided both sides by $m - n$.

But, $m - n = 0$

We can divided any quantity with non- zero quantity only. Here, we divided a quantity with zero quantity.

Hence, the error is in step [5], where we divided both sides by $m - n = 0$, which is not valid.

Example 2.60

Fine the error in the following "Proof":

Let $x, y,$ *and* z be real numbers such that $x = y + z$. Then

Step	Re *ason*
$(1)\ x = y + z$	*Given*
$(2)\ x(x-y) = (y+z)(x-y)$	*Multiply both sidesby* $(x - y)$
$(3)\ x^2 - xy = xy - y^2 + xz - yz$	*Simplify*
$(4)\ x^2 - xy - xz = xy - y^2 - yz$	*Subtract xz from both sides*
$(5)\ x(x-y-z) = y(x-y-z)$	*Takethecommon factors out*
$(6)\ x = y$	*Divideboth sides of* $(x-y-z)$

This shows that any two real numbers are the same. For example, if $x = 9$ *and* $y = 7$, *wetake* $z = 2$.

Conclusion is 9=7.

Solution The error in the proof is in step [6] , where we divided both sides by $x - y - z$.

Since $x = y + z, x - y - z = 0$

We divided by 0, which is not valid.

Example 2.61

Find the errors in the following proof that proves that "if m^2 is positive, then m is positive".

Proof:

Suppose that m^2 is positive. Since the implication "If m is positive, then m^2 is positive" is true, we can conclude that m is positive.

Solution Let $P(m)$ be the proposition "m is positive" and $Q(m)$ be the proposition "m^2 is positive".

Our hypothesis is that $Q(m)$.

The statement "If m is positive, then m^2 is positive" is the statement $6m\,(P(m) \to Q(m))$.

Hence, from the hypothesis $Q(m)$ and the statement $6m\,(P(m) \to Q(m))$, we cannot conclude $P(m)$,since we are not using valid rule of inference.

Thus, the conclusion is a fallacy.

For example, if $m = -1, m^2 = 1$ is positive, but m is negative.

Example 2. 62

Find the error in the following proof:

Theorem: If r is a real number , then r^2 is a positive real number

Proof: Let P_1 be "r is positive" and P_2 be "r is negative" and R be "r^2 is positive"

To show that $P_1 \to R$.

When r is positive, then r^2 is positive, since it is a product of two positive numbers r and r.

To show that $P_2 \to R$.

When r is negative, then r^2 is positive, since it is a product of two negative numbers r and r.

$P_2 \to R$.

This completes the proof.

Solution : Since r is a real number, $r = 0\,(or)\,r > 0\,(or)\,r < 0$.

The error in the proof is that we missed the case $r = 0$.

When $r = 0$, $r^2 = 0$ is not positive.

If P is "r is a real number", then

$P \times P_1 \vee P_2 \vee P_3$, *where* P_1 is "r is positive",

P_2 is "r is negative" and P_3 *is* "$r = 0$"

Also, we have to prove the results for all the three cases P_1, P_2, *and* P_3.

Thus the conclusion is fallacy.

Many incorrect arguments are based on a fallacy called begging the question. This fallacy occurs when a proposition is proved using itself, or a proposition equivalent to it. Hence, this fallacy is also called circular reasoning.

Example 2. 63

Is the following argument correct? "Suppose we have showed that m is an even integer whenever m^2 is an even integer".

Proof:

Suppose that m^2 is an even integer. Then $m^2 = 2u$ for some integer u. Let $m = 2u$ for some integer v,This shows that m is even".

Solution:

This argument is incorrect. In the proof, the statement "let $m = 2u$ for some integer v" occurs .No argument has been given to show this.

Since this statement is equivalent to the statement being proved, this is circular reasoning . Here, the result itself is correct, only the method of proof is wrong.

SUMMARY

*The logic based on the analysis of predicates in any statement is called predicate logic.

*A statement function becomes a statement when the variable is replaced by the name of any object. The statement resulting from a replacement is called a substitution instance of the statement function and is a formula of statement calculus.

*The integer m is even if there exists an integer k such that $m = 2k$ and it is odd if there exists an integer k such that $m = 2k + 1$.

*The real number r is rational if there exist integers a *and* b with $b \neq 0$ such that $r = \frac{a}{b}$. A real number which is not rational is called irrational.

*Many incorrect arguments are based on a fallacy called begging the question.

COMBINATORICS

2.1 Principle of mathematical induction:

Let $P(n)$be a statement or proposition involving for all positive integers n.

Step 1: Basis step: We verify that $P(1)$ is true.

Step 2: Inductive step: We show that for all positive integers k, if $P(k)$is true, then $P(k + 1)$ is true.

Problems:

1. Prove by the principle of mathematical induction, for „n" a positive integer,

$$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution:

Let $P(n)$ be proposition that $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$, where „$n$" is a positive integer.

Step 1: Basis step: $P(1)$: $1^2 = 1 = \frac{6}{6} = \frac{1(1+1)(2(1)+1)}{6}$, therefore $P(1)$ is true.

Step 2: Inductive step: Assume that $P(k)$: $1^2 + 2^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$ is true .... (1)

To prove that$P(k + 1)$ is true.

$k+1)^2$ on both sides in (1), we have

$$1^2+2^2+\ldots+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2.$$

$$=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$=\frac{(k+1)\left(k(2k+1)+6(k+1)\right)}{6}$$

$$=\frac{(k+1)\left(2k^2+k+6k+6\right)}{6}$$

$$=\frac{(k+1)\left(2k^2+4k+3k+6\right)}{6}$$

$$=\frac{(k+1)\left(2k(k+2)+3(k+2)\right)}{6}$$

$$1^2+2^2+\ldots+k^2+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.$$

Therefore $P(k + 1)$ is true.

By the principle of mathematical induction, $1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$ is true for any positive integer n.

2. Prove that 8^n-3^n is a multiple of 5.

Solution:

Let $P(n)$ be proposition that 8^n-3^n is a multiple of 5.

Step 1: Basis step: $P(1)$: $8^1-3^1=5$ is a multiple of 5, therefore $P(1)$ is true.

Step 2: Inductive step: Assume that $P(k)$: 8^k-3^k is true

Therefore $8^k-3^k=5m$, where m is an integer .... (1)

To prove that $P(k + 1)$ is true.

Now, $8^{k+1}-3^{k+1}=8\cdot 8^k-3\cdot 3^k$

$$=5\cdot 8^k+3\cdot 8^k-3\cdot 3^k$$

$$=5\cdot 8^k+3(8^k-3^k)$$

$= 5 \cdot 8^k + 3(5m)$, (Using (1))

$= 5(8^k + 3m)$ is a multiple of 5.

Therefore $P(k + 1)$ is true.

By the principle of mathematical induction, $8^n - 3^n$ is a multiple of 5.

3. Prove that $a^n - b^n$ is a multiple of 5.

Solution:

Let $P(n)$ be proposition that $a^n - b^n$ is a multiple of 5.

Step 1: Basis step: $P(1)$: $a^1 - b^1 = a - b$ is a multiple of 5, therefore $P(1)$ is true.

Step 2: Inductive step: Assume that $P(k)$: $a^k - b^k$ is true

Therefore $a^k - b^k = 5(a-b)$, where m is an integer

$a^k = 5(a-b) + b^k$ .... (1)

To prove that $P(k + 1)$ is true.

Now, $a^{k+1} - b^{k+1} = a \cdot a^k - b \cdot b^k$

$= a \cdot (5(a-b) + b^k) - b \cdot b^k$, (Using (1))

$= 5a(a-b) + a \cdot b^k - b \cdot b^k$

$= (a-b)(5a + b^k)$ is a multiple of $a - b$.

Therefore $P(k + 1)$ is true.

By the principle of mathematical induction, $a^n - b^n$ is a multiple of $a - b$.

4. Show that $2^n < n!$ for all $n \geq 4$.

Solution:

Let $P(n)$ be proposition that $2^n < n!$ for all $n \geq 4$.

Step 1: Basis step: $P(4)$: $2^4 = 16 < 24 = 4!$, therefore $P(4)$ is true.

Step 2: Inductive step: Assume that $P(k)$: $2^k < k!$ is true ....(1)

To prove that $P(k + 1)$ is true.

Now, multiplying $k+1$ on both sides in (1), we have

$$(k+1)2^k < (k+1)k!$$

$2\cdot 2^k < (k+1)!$, since 2 <k + 1, because $k \geq 4$.

$$\Rightarrow 2^{k+1} < (k+1)!.$$

Therefore $P(k + 1)$ is true.

By the principle of mathematical induction, $2^n < n!$ is true for all $n \geq 4$.

5. Prove that for every positive integer n, $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1}-1\right)$.

Solution:

Let $P(n)$ be proposition that $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1}-1\right)$, where „$n$" is a

$$1 > 0.828 = 2\left(\sqrt{1+1}-1\right)$$, therefore $P(1)$ is true.

Step 2: Inductive step: Assume that $P(k)$: $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{k}} > 2\left(\sqrt{k+1}-1\right)$ is true .... (1)

To prove that $P(k + 1)$ is true.

Now, adding $\frac{1}{\sqrt{k+1}}$ on both sides in (1), we have

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > 2\left(\sqrt{k+1}-1\right)+\frac{1}{\sqrt{k+1}}.$$

$$> \frac{2(k+1)+1}{\sqrt{k+1}}-2$$

$$> \frac{2\sqrt{(k+1)+1}\sqrt{(k+1)+1}}{\sqrt{k+1}}-2$$

$$> \frac{2\sqrt{(k+1)+1}\sqrt{k+1}}{\sqrt{k+1}}-2,$$

since $(k+1)+1 > k+1$, because "k" is a positive integer.

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} > 2\left(\sqrt{(k+1)+1}-1\right).$$

Therefore $P(k + 1)$ is true.

By the principle of mathematical induction, $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}>2\left(\sqrt{n+1}-1\right)$ is true for any positive integer n.

6. Use Mathematical Induction, prove that

$$\sum_{m=0}^{n} 3^m = \frac{3^{n+1}-1}{2}$$

Solution:

Let P(n) : $\sum_{m=0}^{n} 3^m = \frac{3^{n+1}-1}{2}$

P(n) : $3^0 + 3^1 + 3^2 + \ldots\ldots 3^n = \frac{3^{n+1}-1}{2}$

Step1: To prove P(1) is true

P(1) is basis step

To prove P(0) is true

For n = 0 P(0) : $3^0 = \frac{3^{0+1}-1}{2}$

$= \frac{3-1}{2}$

$3^0 = \frac{2}{2}$

1 = 1 is true

Step2: Assume P (k) is true

P(k) : $3^0 + 3^1 + 3^2 + \ldots\ldots 3^k = \frac{3^{k+1}-1}{2}$ is true $\rightarrow$ (1)

Step3: Claim P (K+1) is true

(ie) to prove P (K+1) = $\frac{3^{k+1}-1}{2}$

$$P(k+1) = 3^0 + 3^1 + \ldots\ldots + 3^k + 3^{k+1}$$

$$= \frac{3^{k+1}-1}{2} + 3^{k+1} \qquad \text{[using (1)]}$$

$$= \frac{3^{k+1}-1+2\ (3^{k+1})}{2}$$

$$= \frac{3^1(3^{k+1})-1}{2}$$

$$= \frac{3^1.3^{k+1}-1}{2}$$

$$= \frac{3^{k+2}-1}{2}$$

$$P(k+1) = \frac{3^{(k+1)+1}-1}{2}$$

$\therefore$ P(k+1) is true

Step2: Assume P (k) is true

P(k) : $3^0 + 3^1 + 3^2 + \ldots\ldots 3^k = \frac{3^{k+1}-1}{2}$ is true $\rightarrow$ (1)

Step3: Claim P (K+1) is true

(ie) to prove P (K+1) = $\frac{3^{k+1}-1}{2}$

$$P(k+1) = 3^0 + 3^1 + \ldots\ldots + 3^k + 3^{k+1}$$

$$= \frac{3^{k+1}-1}{2} + 3^{k+1} \qquad \text{[using (1)]}$$

$$= \frac{3^{k+1}-1+2\ (3^{k+1})}{2}$$

$$= \frac{3^1(3^{k+1})-1}{2}$$

$$= \frac{3^1.3^{k+1}-1}{2}$$

$$= \frac{3^{k+2}-1}{2}$$

$$P(k+1) \quad = \frac{3^{(k+1)+1}-1}{2}$$

$\therefore$ P(k+1) is true

Step 4 : Conclusion

By the principle of Mathematical induction

$$P(n) \quad : \quad \sum_{m=0}^{n} 3^m = \frac{3^{n+1}-1}{2}$$

is true for n$\geq$ 0.

7. Show that n^3+2n is divisible by 3

Solution :

Let P(n) : n^3+2n is divisible by 3

Step 1 : To prove P (1) is true

For n = 1

P (1) = 1^3+2(1) = 3 is divisible by 3

Step2: Assume that P (k) is true

P (k) = k^3+2k is divisible by 3 is true $\longrightarrow$ (1)

Step 3: Claim P (k+1) is true

P (K+1) = $(k+1)^3$+2(k+1)

P (K+1) = k^3+3k^2+ 3k +1 + 2k + 2

= k^3+3k^2+ 3k + 2k + 3

= $(k^3+2k)+3(k^2+k+1)$ is divisible by 3

Because k^3+2k is divisible by 3 by $\longrightarrow$ (1)

& 3(k^2+k+1) is a multiple of 3

$\therefore$ P (k+1) is true

Step 4: Conclusion

By the principle of mathematical induction P (n)

= n^3+2n is divisible by 3.

8. Prove by Mathematical induction that $6n$+2 + $7n$+1 is divisible by 43 for each positive integer n≥0

Solution:

Let P (n): $6n$+2 + $72n$+1 is divisible by 43

Step1: To prove P (1) is true, Given n≥0

(ie) to prove P (0) is true

For n = 0

P (0) = $6^{0+2} + 7^{2(0)}$ +1

= $6^2 + 7$

= $3^6 + 7$

P(0) = 43 is divisible by 43 is true

Step 2: Assume that P (k) is true

(ie) P (k) = $6^{k+2} + 7^{2k+1}$ is divisible by 43 is true

$6^{k+2} + 7^{2k+1}$ = 43 m (m is an integer)

6^{k+2} = 43 m - $7^{2k+1} \rightarrow (1)$

Step3: Claim P (k+1) is true

P (k+1) = $6^{(k+1)+2} + 7^{2(k+1)+1}$

= $6^{k+2}. 6^1 + 7^{2k+2+1}$

= $6^{k+2}. 6 + 7^{2k+1} . 72$

= $6(6^{k+2}) + 49(7^{2k+1})$

= 6 [43m - 7^{2k+1}] + 49 ($72k$+1) (∴ by (1))

= 6 (43m) -6(7^{2k+1}) + 49 (7^{2k+1})

= 6 (43m) - 43 (7^{2k+1})

= 43 (6m - 7^{2k+1}) is multiple of 43

∴ P (k+1) is divisible by 43

∴ P (k+1) is true

Step 4: Conclusion

By principle of mathematical Induction

P (n) : $6^{n+2} + 7^{2n+1}$ is divisible by 43.

9. Use Mathematical Induction prove that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}} > \sqrt{n} \text{ for } n \geq 2$$

Solution:

Let P (n): $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$ for $n \geq 2$

Step 1: To prove P(2) is true

Given $n \geq 2$

P(2) : $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}} = 1.707 > \sqrt{2} = (1.414)$ is true

Step 2: Assume that P(k) is true

Step2: Assume that P(k) is true

P(k) : $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k}$ is true $\quad \rightarrow (1)$

Step3: Claim P (k+1) is true

(ie) To prove P (k+1) > $\sqrt{k+1}$

P(k+1) : $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

$= \sqrt{k} + \frac{1}{\sqrt{k+1}}$

$= \frac{\sqrt{k}\sqrt{k+1+1}}{\sqrt{k+1}}$

$= \frac{\sqrt{k(k+1)+1}}{\sqrt{k+1}}$

$= \frac{\sqrt{k.k+1}}{\sqrt{k+1}}$

$= \sqrt{k^2} + 1$

$= \frac{\sqrt{k+1}}{\sqrt{k+1}}$

$P(k+1) > \sqrt{k+1}$

$\therefore$ P(k+1) is true

$$P(n) : \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n+1} \text{ is true for } n \geq 2.$$

$$\frac{n(2n-1)(2n+1)}{3}$$

$$\frac{n(n-1)}{2}$$

3) $2 + 2^2 + 2^3 + \ldots\ldots\ldots + 2n = 2^{n+1} - 2$

4) $3^{2n} + 4^{n+1}$ is divisible by 5 for $n \geq 0$

5) $n^3 - n$ is divisible by 5

6) 6) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots. + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

$$\frac{n(n+1)(n+2)(n+3)}{4} \quad \text{Where } n \geq 0$$

8) Show that $n < 2^n$

2.2 Principle of strong induction:

To prove that $P(n)$is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

Basis step: We verify that the proposition $P(1)$is true.

Inductive step: We show that the conditional statement $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k + \mathrm{I})$ is true for all positive integers k.

Strong induction is sometimes called the second principle of mathematical induction or complete induction.

Example:

1. Show that if n is an integer greater than I, then n can be written as the product of primes.

Solution:

Let $P(n)$ be the proposition that n can be written as the product of primes.

Basis step: $P(2)$is true, because 2 can be written as the product of one prime, itself.

Inductive step: Assume that $P(j)$is true for all positive integers j with $j \leq k$, that is, the assumption that j can be written as the product of primes whenever j is a positive integer at least 2 and not exceeding k.

To complete the inductive step, it must be shown that $P(k + \mathrm{I})$ is true under this assumption, that is, that $k + \mathrm{I}$ is the product of primes.

Consider there are two cases.

(i). When $k + 1$ is prime and

(ii). When $k + 1$ is composite.

If $k + \mathrm{I}$ is prime, we immediately see that $P(k + \mathrm{I})$ is true.

Otherwise, $k + \mathrm{I}$ is composite and it can be written as the product of two positive integers a and b with $2 \leq a \leq b < k + \mathrm{l}$.

By the inductive hypothesis, both a and b can be written as the product of primes.

Thus, if $k + \mathrm{I}$ is composite, it can be written as the product of primes, namely, those primes in the factorization of a and those in the factorization of b.

It follows that, if n is an integer greater than I, then n can be written as the product of primes.

The well-ordering property:

Every nonempty set of nonnegative integers has a least element.

Example:

1. For any non – negative integer n, $5n = 0$.

Solution:

Let $P(n)$: $5n = 0$.

Basis step: $5 \cdot 0 = 0$, for $n = 0$.

Inductive step: Suppose that $5j = 0$, for all non-negative integers j with $0 \leq j \leq n$.

Let $k + 1 = I + j$, where $i$and j are natural numbers less than $k + 1$.

By the induction hypothesis, $5(i + j) = 5 \cdot i + 5 \cdot j = 0 + 0 = 0$.

Therefore, for any non – negative integer n, $5n = 0$.

The basics of counting:

The basic counting principles are

i) The product rule and

ii) The sum rule.

Product rule:

If one job can be done in „*m*" ways and another job can be done in „*n*" ways, then the total number of ways in which both the jobs can be done is equal to *mn*.

Sum rule:

If one job can be done in „*m*" ways and another job can be done in „*n*" ways and if there is no way common to both jobs, then the total number of ways in which either of the two jobs can be done is equal to m + n.

Examples:

1. How many different bit strings of length 7 are there?

Solution:

Each of the 7 bits can be chosen in two ways because each bit is either 0 or 1.

Therefore, by product rule, the total number of different bit strings of length 7

$$= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

$$= 2^7$$

$$= 128.$$

2. How many different 8 bit strings are there that begin and end with 1?

Solution:

An eight bit string that begin and end with 1 can be constructed in 6 steps. By selecting 2^{nd} bit, 3^{rd} bit, 4^{th} bit, 5^{th} bit, 6^{th} bit and 7^{th} bit and each bit can be selected in 2 ways.

Therefore the total number of 8-bit strings that begin and end with 1

$$= 2 \times 2 \times 2 \times 2 \times 2 \times 2$$

$$= 2^6$$

$$= 64.$$

3. How many different 8 bit strings are there that end with 111?

Solution:

An eight bit string that end with 111 can be constructed in 5 steps.

By selecting 1^{st} bit, 2^{nd} bit, 3^{rd} bit, 4^{th} bit and 5^{th} bit and each bit can be selected in 2 ways.

Therefore the total number of 8-bit strings that end with 111

$$= 2 \times 2 \times 2 \times 2 \times 2$$

$$= 2^5$$

$$= 32.$$

4. The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution:

The procedure of labeling a chair consists of two tasks, namely, assigning one of the 26 letters and then assigning one of the 100 possible integers to the seat.

The product rule shows that there are $26 \cdot 100 = 2600$ different ways that a chair can be labeled.

Therefore, the largest number of chairs that can be labeled differently is 2600.

5. A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution:

The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways.

By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

6. There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?
7. How many different license plates are available if each plate contains a sequence of three letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?
8. A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution:

The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are

$$23 + 15 + 19 = 57 \text{ ways to choose a project.}$$

9. How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution:

We can construct a bit string of length eight that either starts with a 1 bit or ends with the two bits 00, by constructing a bit string of length eight beginning with a 1 bit or by constructing a bit string of length eight that ends with the two bits 00.

By the product rule, the first bit can be chosen in only one way and each of the other seven bits can be chosen in two ways. Therefore we can construct a bit string of length eight that begins with a 1 in $2^7= 128$ ways.

Similarly, we can construct a bit string of length eight ending with the two bits 00, in $2^6 = 64$ ways. This follows by the product rule, because each of the first six bits can be chosen in two ways and the last two bits can be chosen in only one way.

Some of the ways to construct a bit string of length eight starting with a 1 are the same as the ways to construct a bit string of length eight that ends with the two bits 00. There are $2^5= 32$ ways to construct such a string. This follows by the product rule, because the first bit can be chosen in only one way, each of the second through the sixth bits can be chosen in two ways, and the last two bits can be chosen in one way.

It follows that, the number of bit strings of length eight either start with a 1 bit or end with the two bits 00 = 128 + 64 – 32 = 160.

2.3 The pigeonhole principle:

If k is a positive integer and $k + 1$ or more objects (pigeons) are placed into k boxes (pigeonholes), then there is at least one box containing two or more of the objects.

Proof:

We will prove the pigeonhole principle using a proof by contradiction.

Assume $k + 1$ or more objects are placed into k boxes.

To prove that at least one box containing two or more of the objects.

Suppose that none of the k boxes contains more than one object.

Therefore each and every box has exactly one object.

Since there are n boxes, then the total number of objects would be at most k.

This is a contradiction to our assumption there are at least $k + 1$ objects.

Therefore at least one box containing two or more of the objects.

The generalized pigeonhole principle:

If „m‟ pigeons occupies „n‟ holes ($m > n$), then at least one hole has more than $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ pigeons. Here $[x]$ denotes the greatest integer less than or equal to x, which is a real number.

Proof:

Assume „m‟ pigeon occupies „n‟ holes ($m > n$).

To prove: At least one hole has more than $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ pigeons.

Suppose not, i.e., at least one hole has not more than $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ pigeons.

Each and every hole has exactly $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ pigeons.

Since we have n holes, totally there are $n\left[\left(\frac{m-1}{n}\right) + 1\right]$ pigeons.

$\Rightarrow m - 1 + n$ pigeons.

$\Rightarrow m + n - 1$ pigeons, which is a contradiction to our assumption that there are m pigeons.

Therefore, at least one hole has more than $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ pigeons.

Examples:

1. Show that, among 100 people, at least 9 of them were born in the same month.

Solution:

Here, No. of pigeons = m = No. of people = 100

No. of holes = n = No. of months = 12

Then by generalized pigeonhole principle, we have $\left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{100-1}{12} \right\rfloor + 1 = 9$, were born in the same month.

2. Show that if seven colours are used to paint 50 bicycles, at least 8 bicycles will be the same colour.

Solution:

Here, No. of pigeons = m = No. of bicycle = 50

No. of holes = n = No. of colours = 7

Then by generalized pigeonhole principle, we have

$$\left\lceil \frac{m-1}{n} \right\rceil + 1 = \left\lceil \frac{50-1}{7} \right\rceil + 1 = 8$$

Therefore, at least 8 bicycles will have the same colour.

3. Show that if 25 dictionaries in a library contain a total of 40,325 pages, then one of the dictionaries must have at least 1614 pages.

Solution:

Here, No. of pigeons = m = No. of pages = 40,325

No. of holes = n = No. of dictionaries = 25

Then by generalized pigeonhole principle, we have

$$\left\lceil \frac{m-1}{n} \right\rceil + 1 = \left\lceil \frac{40325-1}{25} \right\rceil + 1 = 1614 \text{ pages.}$$

4. Prove that in any group of six people, there must be at least 3 mutual friends or at least 3 mutual enemies.

Solution:

Let those six people will be A, B, C, D, E and F.

Fix A. The remaining 5 people can be accommodating into 2 groups via.

1. Friends of A and
2. Enemies of A.

Now, by generalized pigeonhole principle, at least one of the group must contain $\left\lceil \frac{m-1}{n} \right\rceil + 1 = \left\lceil \frac{5-1}{2} \right\rceil + 1 = 3$ people.

Let the group friend of A contain 3 people (Let it be B, C, D)

Case – I: If any two of these 3 people (B, C, D) are friends, then these two together with A form 3 mutual friends.

Case – II: If no two of these 3 people (B, C, D) are friends, then these three people are mutual enemies.

In either case, we get the required conclusion.

If the group of enemies of A contains 3 people, by the above similar argument we get the required conclusion.

Therefore, in any group of six people, there must be at least 3 mutual friends or at least 3 mutual enemies.

5. If we select 10 points in the interior of an equilateral triangle of side 1, show that there must be at least 2 points whose distance apart is less than $\frac{1}{3}$.

Solution:

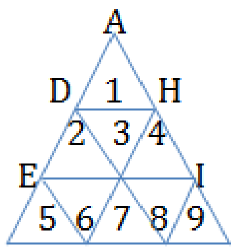
Let ABC be the given equilateral triangle.

Let D and E are the points of trisection of the side AB, F and G are the points of trisection of the side BC, H and I are the points of trisection of the side

AC. Hence the triangle ABC divided into 9 equilateral triangles each of side $\frac{1}{3}$.

Now, No. of pigeons = m = No. of interior points = 10

No. of holes = n = No. of triangles = 9



Then, by generalized pigeonhole principle,

at least one triangle contains

$\left\lceil \frac{m-1}{n} \right\rceil + 1 = \left\lceil \frac{10-1}{9} \right\rceil + 1 = 2$, interior points.

Since each triangles of length $\frac{1}{3}$, the distance between any two interior points of any sub triangle cannot exceeds $\frac{1}{3}$.

Therefore, there must be at least 2 points whose distance apart is less than $\frac{1}{3}$.

6. Find the minimum number of students need to guarantee that five of them belong to the same subject, having major as English, Maths, Physics and Chemistry.

Solution:

No. of pigeon hole = No. of subjects = n = 4.

Let k be the number of students (pigeon) in each subject.

$k + 1 = 5$

$k = 4$

$\therefore$ The total number of students = $kn + 1$ =17.

Permutation with repetition:

The number of permutation of 'n' objects taken 'r' at a time with repetition allow is n^r.

Example: The number of string of length 'n' can be formed from the English

Alphabet is $(26)^n$.

Problems:

1. How many four digit numbers can be formed using the digit 2,4,6,8 when repetition of digit is allowed?

Solution: There are 4 ways of filling each of the digit position.

$\therefore$ Total no. of 4 digits formed = 4.4.4.4

$= (4)^4$

$= 256$

2. Find the number of bit strings of length 10 that either begin with 1 or end with 0

Solution:

Begin with 0 or end with 1, only 2 digit.

Bit string of length n = 10

No. of bit string $= 2^n$

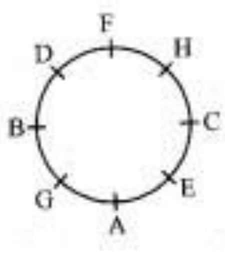
$= 2^{10}$

Circular Permutation:

The argument of objects in a circle is called circular permutation.

The number of arrangement n objects in a circle = (n-1)! .

Example: In how many different ways can five men and five women sit around a table?



Solution: Here n = 10

No. of circular permutation = (n-1)!

= (10-1) !

= 9!

= 9x8x7x6x5x4x3x2x1

= 3,628,80 .

Exercise:

1) Find the number of distinct permutations that can be formed from all the letters of each word

   (i) MATHEMATICS (ii) FORMULA **[Ans: (i)4989600 (ii)5039]**

2) In how many ways can the letters of the following words to be arranged?

   (i) Radio (ii) Foreign **[Ans: (i)120 (ii)5040]**

3) In how many ways can 8 people sit around a table? **[Ans: 5040]**

4) Find n if (i) nP2 = 20 (ii) nP3 = 5nP2 **[Ans: (i) 5 (ii) 7]**

5) A committee of 11 members sit at a round table. In how many ways can they be seated if the „President" and „Secretary" choose to sit together? **[Ans: 2 × 9!]**

6) There are 6 books on Mathematics, 3 on Physics and 2 on Chemistry. In how many ways can these be placed on a shelf if books on the same subject are to be together.

   [Ans: 51840 ways]

2.5 Combination:

An r – combination of elements of a set is an unordered selection of *r* elements from the set. Thus, an r – combination is simply a subset of the set with *r* elements.

***C*(*n, r*):** The number of r – combinations of a set with *n* elements.

Note:

1. The process of arranging things is called permutation.
2. The process of selecting things is called combination.
3. $P(n,\ r) = {}_nP_r = \frac{\text{n!}}{\text{(n - r)!}}$, $C(n,\ r) = {}_nC_r = \frac{\text{n!}}{\text{r!(n - r)!}}$.
4. ${}_nC_n = {}_nC_0 = {}_nP_0 = 1,\ {}_nC_1 = {}_nP_1 = n,\ {}_nC_r = {}_nC_{n-r},\ {}_nP_n = n!,$ $n! = n(n-1)(n-2)\ ...\ 3.2.1\ \&\ 0! = 1! = 1$
5. Pascal"s rule: ${}_nC_r + {}_nC_{n-r} = {}_{n+1}C_r$.

Combinations with repetitions:

The number of *r*-combinations (selection of *r* objects) of n kinds of objects, if repetitions of the objects is allowed = C (n + r – 1, r)

Examples:

1. In how many ways 6 persons occupy 3 vacant seats?

Solution:

The total number of ways = ${}_6P_3$

Since $P(n, r) = {}_nP_r = \dfrac{n!}{(n-r)!}$.

$\therefore$ The total number of ways = $\dfrac{6!}{(6-3)!} = \dfrac{6!}{3!} = \dfrac{6\times5\times4\times3!}{3!} = 120$.

2. In how many ways can letters of the word "INDIA" be arranged?

Solution:

The word "INDIA" contains 5 letters of which 2 are I‟s .

Therefore number of distinct permutations = $\dfrac{5!}{2!} = \dfrac{5\times4\times3\times2!}{2!} = 60$.

3. Find the number of distinct permutations that can be formed from all the letters of each word (i). RADAR and (ii). UNUSUAL.

Solution:

(i) The word "RADAR" contains 5 letters of which 2 R‟s and 2 A‟s.

Therefore the number words possible = $\dfrac{5!}{2!\,2!} = \dfrac{5\times4\times3\times2!}{2!2!} = 30$.

(ii) The word "UNUSUAL" contains 7 letters of which 3 are U‟s.

Therefore the number words possible = $\dfrac{7!}{3!} = \dfrac{7\times6\times5\times4\times3!}{3!} = 840$.

4. Find the number of ways in which 5 boys and 5 girls can be seated in a row if the boys and girls are to have alternate seats.

Solution:

Case (i): Boys can be arranged among themselves in 5! ways.

___B___B___B___B___B___

There are six places for girls.

Hence there are ${}_6P_5 \times 5!$ ways.

Case (ii): Girls can be arranged among themselves in 5! ways.

___G___G___G___G___G___

There are six places for boys.

Hence there are $_6P_5 \times 5!$ ways.

Therefore, by sum rule, taking 2 cases into account, there are $2 \times {}_6P_5 \times 5!$ ways.

Therefore the total number of arrangements = $2 \times \frac{6!}{(6-5)!} \times 5! = 2 \times 720 \times 120 = 172800$.

5. In how many ways can 5 persons be selected from amongst 10 persons?

Solution:

The selection can be done in $_{10}C_5$ ways.

Therefore number of ways = $\frac{10!}{(10-5)! \times 5!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5! \times 5!} = \frac{30240}{120} = 252$.

6. How many ways are there to form a committee, if the committee consists of 3 educationalist and 4 socialist, if there are 9 educationalist and 11 socialist.

Solution:

The 3 educationalist can be chosen from 9 educationalist in $_9C_3$ ways.

The 4 socialist can be chosen from 11 socialist in $_{11}C_4$ ways.

Therefore number of ways = $\frac{10!}{(10-5)! \times 5!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5! \times 5!} = \frac{30240}{120} = 252$.

Therefore, by product rule, the number of ways to select the committee

$$= {}_9C_3 \times {}_{11}C_4$$

$$= \frac{9!}{(9-3)! \times 3!} \times \frac{11!}{(11-4)! \times 4!}$$

$$= \frac{9 \times 8 \times 7 \times 6!}{6! \times 3!} \times \frac{11 \times 10 \times 9 \times 8 \times 7!}{7! \times 4!}$$

$$= 84 \times 330$$

= 27720 ways.

7. A box contains six white balls and five red balls. Find the number of ways four balls can be drawn from the box if (i). They can be any colour.

(ii). Two must be white and 2 red.

(iii). They must all be the same colour.

Solution:

(i). Four balls of any colour can be chosen from 6 + 5 = 11 balls in $_{11}C_4$ ways

$$= \frac{11\times10\times9\times8}{1\times2\times3\times4} = 330.$$

(ii). The two white balls can be chosen in $_6C_2$ ways.

The two red balls can be chosen in $_5C_2$ ways.

Number of ways selecting four balls 2 must be white and 2 must be red

$$= {}_6C_2 \times {}_5C_2 = \frac{6\times5}{1\times2}\times\frac{5\times4}{1\times2}$$

= 15 × 10 = 150 ways.

(iii). Number of ways selecting four balls and all of same colour

$$= {}_6C_4 + {}_5C_4 = {}_6C_2 + {}_5C_1$$

$$= \frac{6\times5}{1\times2} + \frac{5}{1}$$ = 15 + 5 = 20 ways.

8. Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members form the mathematics department and four from the computer science department?

Solution:

The 3 faculty members can be chosen from 9faculty members in the mathematics department in $_9C_3$ ways.

The 4 faculty members can be chosen from 11faculty members in the computer science department in $_{11}C_4$ ways.

Therefore, by product rule, the number of ways to select the committee

$$= {}_9C_3 \times {}_{11}C_4$$

$$= \frac{9!}{(9-3)!\times3!} \times \frac{11!}{(11-4)!\times4!}$$

$$= \frac{9\times8\times7\times6!}{6!\times3!} \times \frac{11\times10\times9\times8\times7!}{7!\times4!}$$

$$= 84 \times 330 = 27720 \text{ ways.}$$

9. From a committee consisting of 6 men and 7 women, in how many ways can be select a committee of

(a) 3 men and 4 women.

(b) 4 members which has atleast one women.

(c) 4 persons that has atmost one man.

(d) 4 persons of both sexes.

(e) 4 person in which Mr. and Mrs. Kannan are not included.

Solution:

Committee consisting of 6 men and 7 women

a) 3 men can be selected from 6 men in $6C_3$ ways

4 women can be selected from 7 women in $7C_4$ ways

By product rule, the committee of 3 men and 4 women

can be selected in $= 6C_3 \text{ x } 7C_4$ ways

$$= \frac{6x5x4}{1x2x3} X \frac{7x6x5x4}{1x2x3x4}$$

$= 20 \text{ x } 35$

$= 700$ ways

b) Form the committee of atleast one women, we have the following possibilities

i) 1 women and 3 men

ii) 2 women and 2 men

iii) 3 women and 1 men

iv) 4 women and 0 men

∴ The selection can be done in

$= (7C_1 \text{ x } 6C_3) + (7C_2 \text{ x } 6C_2) + (7C_3 \text{ x } 6C_1) + (7C_4 \text{ x } 6C_0)$ ways

$$= [7x \frac{6x5x4}{1x2x3}] + [\frac{7x6}{1x2} X \frac{6x5}{1x2}] + [\frac{7x6x5}{1x2x3} X\ 6] + [\frac{7x6x5x4}{1x2x3x4} X 1]\ [7 \text{ x}$$

$= (7x20) + (21x15) + (35x6) + (35x1)$

$= 140 + 315 + 210 + 35 = 700$ ways

c) For the committee of atmost one men we have the following possibilities

i) 1 men and 3 women

ii) 0 men and 4 women

∴ The selection can be done in

$= (6C_1 \times 7C_3) + (6C_0 \times 7C_4)$ ways

$= (6 \times 35) + (1 \times 35)$

$= 245$ ways

d) For the committee of both sexes, we have the following possibilities

i) 1 men and 3 women

ii) 2 men and 2 women

iii) 3 men and 1 women

∴ The selection can be done in

$= (6C_1 \times 7C_3) + (6C_2 \times 7C_2) + (6C_3 \times 7C_1)$ ways

$= (6 \times 35) + (15 \times 21) + (20 \times 7)$

$= 210 + 315 + 140$

$= 665$ ways

e) Since the committee does not consists of Mr and Mrs. Kannan, we have 5 men and 6 women in the committee

Now we can select 4 members from 11 members

$= 11\,C_4$ ways

$= \dfrac{11!}{4!(7!)}$

$= 330$ ways.

10. How many bit strings of length 10 contain

(1) Exactly 4 1's

(2) Atmost 4 1's

(3) Atleast 4 1's

(4) An equal number of 0's and 1's

Solution:

(1) Exactly 4 1's

A bit string of length 10 can be considered to have 10 positions these 10 positions should be filled with four 1's and six 0's

$\therefore$ No. of required bit strings $= \dfrac{10!}{4!(6!)}$

$= \dfrac{3{,}628{,}800}{(24)(720)}$

= 210 ways

(2) Atmost 4 1's

The 10 positions should be filled with

i) No "1's " and 10 "0's"

ii) 1 "1's " and 9 "0's"

iii) 2 "1's " and 8 "0's"

iv) 3 "1's " and 7 "0's"

v) 4 "1's " and 6 "0's"

$\therefore$ Required no. of bit strongs

$$= \frac{10!}{0!10!} + \frac{10!}{1!9!} + \frac{10!}{2!8!} + \frac{10!}{3!7!} + \frac{10!}{4!6!}$$

= 1 + 10 + 45 + 120 + 210

= 386 ways

(3) Atleast 4 1's

The 10 positions are to be filled up with

i) 4 "1's " and 6 "0's"

ii) 5 "1's " and 5 "0's"

iii) 6 "1's " and 4 "0's"

iv) 7 "1's " and 3 "0's"

v) 8 "1's " and 2 "0's"

vi) 9 "1's " and 1 "0's"

vii) 10 "1's " and 0 "0's"

∴ Required no. of bit strings

$$= \frac{10!}{4!6!} + \frac{10!}{5!5!} + \frac{10!}{6!4!} + \frac{10!}{7!3!} + \frac{10!}{8!2!} + \frac{10!}{9!1!} + \frac{10!}{10!0!}$$

= 210 + 252 + 210 + 120 + 45 + 10 + 1

= 848 ways

(4) An equal no. of 0's and 1's

The 10 positions are to be filled up with five 1's five 0's

∴ Required no. of bit strings

$$= \frac{10!}{5!5!}$$

= 252 ways

EXERCISE:

1) Find the value of

(a) 5C2 **[Ans: 10]**

(b) 10C2 **[Ans: 210]**

(c) 30C28 **[Ans: 435]**

2) In how many ways can a committee of 8 persons be formed out of 10 members?

[Ans: 45]

3) In how many ways can 20 articles be packed in the three parcels so that the first contains 8 articles, the second 7 and the third 5? **[Ans: $\frac{20!}{8!7!5!}$]**

4) There are 6 men and 5 women in a room. Find the number of ways 4 persons can be drawn from the room if

(i) They can be male or female

(ii) Two must be men and two women

(iii) They must all are of the same sex **[Ans: (i)330 (ii)150 (iii)20]**

5) A committee of 12 is to be selected from 10 men and 10 women. In how many wayscan the selection be carried out if

(i) There are no restrictions? **[Ans:1,25,970]**

(ii) There must be equal number of men and women? **[Ans: 44,100]**

(iii) There must be an even number of women? **[Ans: 63,900]**

(iv) There must be more women than men? **[Ans: 40,935]**

(v) There must be atleast 8 men? **[Ans: 10,695]**

6) How many bit strings of length 12 contain

(i) Exactly three 1"s? (ii) Atmost three 1"s?

(iii) Atleast three 1"s? (iv) An equal number of 0"s and 1"s

[Ans: (i)220 (ii)299 (iii)4017 (iv)924]

2.6 Recurrence relation:

A recurrence relation (difference equation) for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, a_o, a_1, ... ,a_{n-1}, for all integers n with $n \geq n_o$. where n_o is a nonnegative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form $a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} + ... + c_ka_{n-k}$, where $c_1, c_2, c_3, \ldots, c_k$ are real numbers and $c_k \neq 0$.

A linear non – homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form $a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} + ... + c_ka_{n-k} + F(n)$, where $c_1, c_2, c_3, \ldots, c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n. Here, the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} + ... + c_ka_{n-k}$ is called associated homogeneous recurrence relation.

Solution of Linear Homogeneous Recurrence Relations with Constant Coefficients:

1. Writer down the characteristic equation for the given recurrence relation. Here, the degree of characteristic equation is one less than the number of terms in recurrence relation.

   Let $a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} + ... + c_ka_{n-k}$ be the recurrence relation.
   Then the characteristic equation is $r^k - c_1r^{k-1} - c_2r^{k-2} - c_3r^{k-3} - ... - c_{k-1}r - c_k = 0$.

2. By solving the characteristic equation find out the characteristic roots.

3. Depends upon the nature of roots, find out the solution a_n as follows:

   (i) Let the roots be real and distinct say $r_1, r_2, r_3, ..., r_k$. Then $a_n = \alpha_1r_1^n + \alpha_2r_2^n + \alpha_3r_3^n + ... + \alpha_kr_k^n$, where $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k$ are arbitrary constants.

(ii) Let the roots be real and equal say $r_1 = r_2 = r_3 = ... = r_k = r$. Then $a_n = \alpha_1 r^n + n\alpha_2 r^n + n^2\alpha_3 r^n + ... + n^{k-1}\alpha_k r^n$, where $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k$ are arbitrary constants.

(iii) When the roots are complex conjugate, then $a_n = r^n(\alpha_1 \cos n\theta + \alpha_2 \sin n\theta)$

4. Apply initial conditions and find out arbitrary constants.

Solution of Linear Non-homogeneous Recurrence Relations with Constant Coefficients:

The solution of non-homogeneous recurrence relation is the sum of two solutions.

(1) Solution of associated homogeneous recurrence relation.

(2) Particular solution depending on F(n) of the given recurrence relation.

Case 1:

When F(n) = $a_0 + a_1 n + a_2 n^2 + ... + a_r n^r$, then the particular solution is of the form $c_0 + c_1 n + c_2 n^2 + ... + c_r n^r$.

Substitute $c_0 + c_1 n + c_2 n^2 + ... + c_r n^r$ in the place of a_n, $c_0 + c_1(n-1) + c_2(n-1)^2 + ... + c_r(n-1)^r$ in the place of a_{n-1}, etc in the given recurrence relation and hence find $c_0, c_1, c_2, ..., c_r$ (equating coefficients)

Case 2:

(i) When F(n) = a^n, then the particular solution is of the form ca^n, if a (base of F(n))is not a root of the characteristic equation. Substitute ca^n in the place of a_n, ca^{n-1} in the place of a_{n-1}, etc in the given recurrence relation and hence find c.

(ii) When F(n) = a^n, then the particular solution is of the form cna^n, if a (base of F(n))is a simple root of the characteristic equation. Substitute cna^n in the place of a_n, $c(n-1)a^{n-1}$ in the place of a_{n-1}, etc in the given recurrence relation and hence find c.

(iii) When F(n) = a^n, then the particular solution is of the form $cn^m a^n$, if a (base of F(n)) is a root of multiplicity m of the characteristic equation. Substitute $cn^m a^n$ in the place of a_n, $c(n-1)^m a^{n-1}$ in the place of a_{n-1}, etc in the given recurrence relation and hence find c.

Examples:

1. Find the recurrence relation for the Fibonacci sequence.

Solution:

The Fibonacci sequence is 0, 1, 1, 2, 3, 5, 8, 13 …

Here, $a_0 = 0, a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3 ...$

$\Rightarrow a_2 = a_1 + a_0, a_3 = a_2 + a_1, a_4 = a_3 + a_2 ...$

Therefore $a_n = a_{n-1} + a_{n-2}$

$\Rightarrow a_n - a_{n-1} - a_{n-2} = 0$ with $a_0 = 0, a_1 = 1$ is the required recurrence relation for the Fibonacci sequence.

2. Find the recurrence relation for $S(n) = 6(-5)^n, n \geq 0$.

Solution:

Given $S(n) = 6(-5)^n, n \geq 0$.

$\therefore S(n-1) = 6(-5)^{n-1}$

$$= \frac{6(-5)^n}{-5}$$

$$S(n-1) = \frac{S(n)}{-5}$$

$$-5S(n-1) = S(n)$$

$\therefore S(n) + 5S(n-1) = 0, n \geq 0$ with $S(0) = 6$.

3. Find the recurrence relation which satisfy $y_n = A3^n + B(-4)^n$.

Solution:

Given $y_n = A3^n + B(-4)^n$ ……………………………………… (1)

$\therefore y_{n+1} = A3^{n+1} + B(-4)^{n+1}$

$y_{n+1} = 3A3^n - 4B(-4)^n$ ……………………………………… (2)

$y_{n+2} = A3^{n+2} + B(-4)^{n+2}$

$y_{n+2} = 9A3^n + 16B(-4)^n$ ……………………………………… (3)

$\therefore$ (3) + (2) – 12(1) $\Rightarrow y_{n+2} + y_{n+1} - 12y_n$

$$= 9A3^n + 16B(-4)^n + 3A3^n - 4B(-4)^n - 12A3^n - 12B(-4)^n$$

Hence $y_{n+2} + y_{n+1} - 12y_n = 0$ is the required recurrence relation.

4. Find the recurrence relation for the sequence $a_n = 2n+9, n \geq 1$.

Solution:

Given $a_n = 2n+9, n \geq 1$.

$\therefore a_{n-1} = 2(n-1)+9$

$a_{n-1} = 2n - 2 + 9$

$a_{n-1} = a_n - 2$

$\therefore a_n = a_{n-1} + 2, n \geq 1$ with $a_0 = 9$ is the required recurrence relation.

5. Solve the recurrence relation y(k) – 8y(k – 1) + 16y(k – 2) = 0 for k ≥ 2, where y(2) = 16 and y(3) = 80.

Solution:

Given y(k) – 8y(k – 1) + 16y(k – 2) = 0 for k ≥ 2, where y(2) = 16 and y(3) = 80.

The above equation can be written as y(n) – 8y(n– 1) + 16y(n – 2) = 0 for n ≥ 2.

$\therefore$ The characteristic equation is $r^2 - 8r + 16 = 0$.

$\Rightarrow (r-4)^2 = 0 \Rightarrow r = 4,4$.

$\therefore$ The solution is $y(n) = \alpha_1 r^n + \alpha_2 n r^n$.

$\Rightarrow y(n) = \alpha_1 4^n + \alpha_2 n 4^n$

Put $n = 2 \Rightarrow y(2) = \alpha_1 4^2 + \alpha_2 2 \cdot 4^2$

$\Rightarrow 16 = 16\alpha_1 + 32\alpha_2$

$\Rightarrow \alpha_1 + 2\alpha_2 = 1$ … .... (1)

Put $n = 3 \Rightarrow y(3) = \alpha_1 4^3 + \alpha_2 3 \cdot 4^3$

$\Rightarrow 80 = 64\alpha_1 + 192\alpha_2$

$\Rightarrow 4\alpha_1 + 12\alpha_2 = 5$ ……………………… ... (2)

$(1)\times 4 \Rightarrow 4\alpha_1 + 8\alpha_2 = 4$ . ... (3)

Subtracting (2) and (3), we have

$4\alpha_2 = 1 \Rightarrow \alpha_2 = \frac{1}{4}$

Sub.: $\alpha_2 = \frac{1}{4}$ in (1), we have

$$\alpha_1 + \frac{1}{2} = 1 \Rightarrow \alpha_1 = \frac{1}{2}$$

$\therefore$ The solution is $y(n) = \frac{1}{2}4^n + \frac{1}{4}n4^n \qquad y(k) = \frac{1}{4}\left(2\cdot 4^k + k4^k\right).$

6. Solve: $a_k = 3a_{k-1}$, for $k \geq 1$, with $a_0 = 2$.

Solution:

Given $a_k = 3a_{k-1}, k \geq 1$ with $a_0 = 2$.

The above equation can be written as $a_n - 3a_{n-1} = 0, n \geq 1.$

$\therefore$ The characteristic equation is $r - 3 = 0 \Rightarrow r = 3$.

$\therefore$ The solution is $a_n = \alpha_1 r^n$.

$\Rightarrow a_n = \alpha_1 3^n$.

Put $n = 0 \Rightarrow a_0 = \alpha_1 3^0 \Rightarrow \alpha_1 = 2$.

$\therefore$ The solution is $a_n = 2\cdot 3^n$ (or) $a_k = 2\cdot 3^k$.

7. Solve the recurrence relation $a_{n+1} - a_n = 3n^2 - n$, $n \geq 0$ and $a_0 = 3$.

Solution:

Given non-homogeneous equation is $a_{n+1} - a_n = 3n^2 - n$, $n \geq 0$and $a_0 = 3$.

The associated homogeneous linear equation is $a_{n+1} - a_n = 0$.

$\therefore$ The characteristic equation is $r - 1 = 0$.

$\Rightarrow r = 1.$

$\therefore$ The solution is $a_n^{(h)} = \alpha_1 r^n$.

$\Rightarrow a_n^{(h)} = \alpha_1 1^n = \alpha_1$.

To find particular solution:

Since the L.H.S of the given recurrence relation is $3n^2 - n$, the solution is of the form $an^3 + bn^2 + cn$.

That is, replace a_n by $an_3 + bn_2 + cn$ and a_{n+1} by $a(n+1)_3 + b(n+1)_2 + c(n+1)$

$\therefore$ The given recurrence relation becomes

$$a(n+1)^3 + b(n+1)^2 + c(n+1) - (an^3 + bn^2 + cn) = 3n^2 - n$$

$$a(n^3 + 3n^2 + 3n + 1) + b(n^2 + 2n + 1) + c(n+1) - an^3 - bn^2 - cn = 3n^2 - n$$

$$an^3 + 3an^2 + 3an + a + bn^2 + 2bn + b + cn + c - an^3 - bn^2 - cn = 3n^2 - n$$

$$n^2(3a) + n(3a + 2b) + a + b + c = 3n^2 - n$$

Equating the coefficients of n^2, n and constant term on both sides, we get

$$3a = 3 \Rightarrow a = 1$$

$$3a + 2b = -1 \Rightarrow 3 + 2b = -1 \Rightarrow b = -2$$

$$a + b + c = 0 \Rightarrow 1 - 2 + c = 0 \Rightarrow c = 1$$

$\therefore$ Particular solution is $a_n^{(p)} = n^3 - 2n^2 + n = n(n^2 - 2n + 1) = n(n-1)^2$

Hence general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$= \alpha_1 + n(n-1)^2$$

8. Solve the recurrence relation S(n) – 3S(n – 1) – 4S(n – 2) = 4^n.

Solution:

Given non-homogeneous equation can be written as $a_n - 3a_{n-1} - 4a_{n-2} = 4^n$.

The associated homogeneous linear equation is $a_n - 3a_{n-1} - 4a_{n-2} = 0$.

$\therefore$ The characteristic equation is $r^2 - 3r - 4 = 0$.

$\Rightarrow (r-4)(r+1) = 0$.

$\Rightarrow r = 4, -1.$

$\therefore$ The solution is $a_n^{(h)} = \alpha_1 r_1^n + \alpha_2 r_2^n$

$$\Rightarrow a_n^{(h)} = \alpha_1 4^n + \alpha_2 (-1)^n.$$

To find particular solution:

Since F(n) = 4^n, the base of F(n) is 4 and 4 is the simple root of the characteristic equation.

Therefore the solution is of the form $cn4^n$.

That is, replace a_n by $cn4^n$, a_{n-1} by $c(n-1)4^{n-1}$ and a_{n-2} by $c(n-2)4^{n-2}$

$\therefore$ The given recurrence relation becomes

$$cn4^n - 3c(n-1)4^{n-1} - 4c(n-2)4^{n-2} = 4^n$$

Dividing by 4^{n-2}, we have

$$16cn - 12c(n-1) - 4c(n-2) = 16$$

$$16cn - 12cn + 12c - 4cn + 8c = 16$$

$$20c = 16$$

$$\therefore c \quad \frac{16}{20}$$

$\therefore$ Particular solution is $a_n^{(p)} = cn4^n = \frac{16}{20} n4^n$.

Hence general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$\therefore a_n = \alpha_1 4^n + \alpha_2 (-1)^n + \frac{4}{5} n4^n.$$

9. Find the solution to the recurrence relation

$a_n = 6a_{n-1}, - 11a_{n-2} + 6a_{n-3}$ with the initial conditions

$a_0 = 2$ and $a_1 = 5$ and $a_2 = 15$

Solution:

Given the recurrence relation

$a_n = 6a_{n-1}, - 11a_{n-2} + 6a_{n-3}$

Step1: The recurrence relation can be written as

$a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

The characteristic equation is

$r^3 - 6r^2 + 11r - 6 = 0$

Step2: To solve characteristic equation waged characteristic root

$r^3 - 6r^2 + 11r - 6 = 0$

put r = 1 $\Rightarrow$ 1 -6+11=0

$\therefore$ r = 1 is a root

By synthetic division

$$
\begin{array}{c|cccc}
1 & 1 & -6 & 11 & -6 \\
 & 0 & 0 & -5 & 6 \\
\hline
 & 1 & -5 & 6 & \boxed{0}
\end{array}
$$

Other roots are given by

$r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$r - 2 = 0$	$r - 3 = 0$
$r = 2$	$r = 3$

$\therefore$ The characteristic roots are r1 = 1, r2 = 2, r3 = 3

Three roots are real and distinct.

Step 3: The solution is

$an = \alpha_1 . r_1^n + \alpha_2 .(r_2^n) + \alpha_3 r_3^n$

$an = \alpha_1 .1^n + \alpha_2 .(2)^n + \alpha_3 .3^n \qquad \rightarrow (A)$

Step4: Given the initial condition is

$a_0 = 2,\ a_1 = 2,\ a_2 = 15$

To find α_1, α_2. & α_3 value

(i) Given $a_0 = 2$, put n = 0 in equation (A) weget

$a_0 = \alpha_1 .1^0 + \alpha_2 .2^0 + \alpha_3 .3^0$

$2 = \alpha_1 + \alpha_2 + \alpha_3$

(ie) $\alpha_1 + \alpha_2 + \alpha_3 = 2 \qquad \rightarrow (1)$

(ii) Given $a_1 = 5$, put n = 1 in equation (A) we get

$a_1 = \alpha_1 .1^1 + \alpha_2 .2^1 + \alpha_3 .3^1$

$5 = \alpha_1 + 2\alpha_2 + 3\alpha_3$

(ie) $\alpha_1 + 2\alpha_2 + 3\alpha_3 = 5 \qquad \rightarrow (2)$

(iii) Given $a_1 = 15$, put n = 2 in equation (A) we get

$a_1 = \alpha_1 .1^1 + \alpha_2 .2^2 + \alpha_3 .3^3$

$15 = \alpha_1 + 4\alpha_2 + 9\alpha_3$

(ie) $\alpha_1 + 4\alpha_2 + 9\alpha_3 = 15 \qquad \rightarrow (3)$

To solve equation 1, 2 & 3

$\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = 2$

Substitute $\alpha 1$, α_2 & α_3 value in equation (A) we get

(A) $\Rightarrow a_n = \alpha_1 \,.\, 1^n + \alpha_2 .\, 2n + \alpha_3 .\, 3n$

$an = 1(1^n) + (-1)\, 2^n + 2(3)^n$

$an = 1^n - 2^n + 2.3^n$

Which is the required solution.

10. Find an explicit formula for the Fibonacci sequence

Solution:

Step1: Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13.......... satisfies the recurrence relation

$f_n = f_{n-1} + f_{n-2}$

(or) $f_n - f_{n-1} - f_{n-2} = 0$

and also satisfies the initial conditions $f_0 = 0$ and $f_1 = 1$

$\therefore$ The characteristic equations $r^2 - r - 1 = 0$

Step 2 : To find characteristic root

$r^2 - r - 1 = 0$

$r = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$ $[\therefore ax^2 - bx + C = 0]$

$r = \dfrac{1 \pm \sqrt{1+4}}{2}$ $r \quad \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$r \quad \dfrac{1 \pm \sqrt{5}}{2}$ Here a = 1, b = -1, c = -1

$r \quad \dfrac{1+\sqrt{5}}{2},$ $r \quad \dfrac{1-\sqrt{5}}{2},$

Two roots are real & distinct

Step 3 : The solution is

$f_n = \alpha_1 r_1^{\mathrm{n}} + \alpha_2 r_2^{\mathrm{n}}$

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1+\sqrt{5}}{2}\right)^n \quad \rightarrow \text{(A)}$$

Step 4 : To find α_1 & α_2

Given initial conditions are $f_0 = 0$ & $f_1 = 1$

(i) Given $f_0 = 0$

Put n = 0 in equation (A) we get

$$f_0 = \alpha_n\left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_n\left(\frac{1-\sqrt{5}}{2}\right)^0$$

$0 = \alpha_1 + \alpha_2$

(ie) $\alpha_1 + \alpha_2 = 0$ $\rightarrow$ (1)

(ii) Given $f_1 = 1$

put n = 1 in equation (A) we get

$$f_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2\left(\frac{1+\sqrt{5}}{2}\right)^1$$

$$f_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) \rightarrow (2)$$

Solving equation 1 & 2

$$2 \Rightarrow \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$1X\left(\frac{1+\sqrt{5}}{2}\right) \Rightarrow \qquad \frac{\alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 0}{\alpha_1\left(\frac{1-\sqrt{5}}{2}\right) - \alpha_2\left(\frac{1+\sqrt{5}}{2}\right) = 1}$$

$$\frac{\alpha_2}{2} = (1-\sqrt{5} - 1 - \sqrt{5}) = 1$$

$$\frac{\alpha_2}{2} = (-2\sqrt{5}) = 1$$

$$\alpha_2 = \left(\frac{1}{-\sqrt{5}}\right)$$

Substitute $\alpha_2 = -\frac{1}{\sqrt{5}}$ in equation (1)

(1) $\Rightarrow \alpha_1 + \alpha_2 = 0$

$$\alpha_1 - \frac{1}{\sqrt{5}} = 0$$

$$\alpha_1 = \frac{1}{\sqrt{5}}$$

Substitute α_1 & α_2 in equation (A) we get

The required solution is

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

11. Solve S(K) - 5S(K-1) + 6S (K-2) = 2 with S(0) = 1, S (1) = -1

Solution:

Given non homogeneous equation is

S(K) - 5S(K-1) + 6S (K-2) = 2

(ie)S_k -5 S_{K-1} +6 S_{K-2} = 2

Here F (n) = 2

Now its associated homogenous equation is

S_k -5 S_{K-1} +6 S_{K-2} = 0

∴ The characteristic equation is

r^2 - 5 r + 6 = 0

$(r-2)\ (r-3) = 0$

$r - 2 = 0$	$r - 3 = 0$
$r = 2$	$r = 3$

∴ The roots are $r_1 = 2, r_2 = 3$

The roots are real & distinct

∴ The solution is $S_k^{(h)} = \alpha_1\, r_1^k + \alpha_2\, r_2^k$

$S_k^{(h)} = \alpha_1\, 2^k + \alpha_2\, 3^k$

To find particular solution

As RHS of the Recurrence Relation is constant

Then the solution is of the form C. where C is a constant

$S_k^{(P)} = C$

$\Rightarrow S_k - 5\, S_{K-1} + 6\, S_{K-2} = 2$ Replace $S_k = C$

$C - 5C + 6C = 2$ $S_{K-1} = C$

$2C = 2$ $S_{K-2} = C$

$C = 1$

$\therefore$ The particular solution is

$$S_k^{(P)} = 1$$

$\therefore$ General solution is

$$S_k = S_k^{(h)} + S_k^{(P)}$$

$S_k = \alpha_1\, 2^k + \alpha_2\, 3^k + 1$ $\rightarrow$ (A)

To find $\alpha 1$ & $\alpha 2$ value

Given Initial Conditions S (0) = 1, S (1) = -1

(i) Given S(0) = 1 (ie) $S_0 = 1$

put K = 0 in equation (A) we get

$S_0 = \alpha_1 .2^o + \alpha_2\, 3^o + 1$

$1 = \alpha_1 + \alpha_2 + 1$

$1\text{-}1 = \alpha_1 + \alpha_2$

(ie) $\alpha_1 + \alpha_2 = 0$ $\rightarrow$ (1)

(ii) Given S(1) = -1 (ie) $S_1 = -1$

Put K = 1 in equation (A) we get

$S_1 = \alpha_1 .2^1 + \alpha_2 .3^1 + 1$

$-1 = 2\alpha_1 + 3\alpha_2 = 1$

$2\alpha_1 + 3\alpha_2 = -2$ $\rightarrow$ (2)

Solving equation 1 & 2 we have

$\alpha_1 = 2,\ \alpha_2 = -2$

Substitute α_1 & α_2 value in equation (A) we get

The solution is

$S_K = 2.\ 2^k - 2.3^k + 1$

(ie) $S_k = 2.\ 2^k - 2.3^k + 1$

3) Find the recurrence relation from $y_n = A\,.2^n + B.3^n$

4) Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - 3a_{n-3}$ given that

$a_0 = 5$, $a_1 = 9$ and $a_2=5$ **[Ans: a_n =5 $(-1)^n$+12n $(-1)^n$-8n2 $(-1)^n$]**

5) Solve the recurrence relation $a_{n+2} - 6a_{n+1} + 9a_n = 0$ with $a_0 = 1$ & $a_1 = 4$

[Ans: $a_n = (1+\frac{n}{3})3^n$

6) Find all the solution of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}\ 7^n$

[Ans: $a_n = a_1 (2^n) + a_2 (3^n) + \frac{79}{20} (7^n)$]

7) Solve S(n)-5(n-1)+6S(n-2)=2n+3n

[Ans: $S(n) = A.2^n + B.3^{2n} - n2^{n+1} + \frac{3}{4}(2n+7)$]

8) Solve S(k)+5S(k-1)=9, S(0)=6 **[Ans: $S(k)= \frac{9}{2}(-5)^k + \frac{3}{2}$]**

9) Solve the recurrence relation $a_{n+2} - 6\,a_{n+1} + 9\,a_n = 3(2^n) + 7(3^n)$, $n \geq 0$ given

that $a_0 = 1$ and $a_1 = 4$ **[Ans: $an = \frac{5}{18} n3^n + 2^n + \frac{7}{18} n^2 3^n$]**

2.7 Generating function:

The generating function for the sequence $a_o, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_o + a_1 x + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k .$$

Examples:

1. Write the generating function for the sequence 1, a, a^2, a^3, a^4, …

Solution:

Given sequence is 1, a, a^2, a^3, a^4…

Generating function is $G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$$\therefore G(x) = 1 + ax + a^2x^2 + a^3x^3 + \ldots$$

$$\Rightarrow G(x) = 1 + ax + (ax)^2 + (ax)^3 + \ldots$$

$\Rightarrow G(x) = (1-ax)^{-1}$, for $|ax| < 1$, since $(1-x)^{-1} = 1+x+x^2+x^3+\dots, |x|<1$.

$\Rightarrow G(x) = \dfrac{1}{1-ax}$, for $|ax|<1$.

2. Find the generating function for the sequence $2^0, 2^1, 2^2 \dots 2^r$.

Solution:

Given sequence is $2^0, 2^1, 2^2 \dots 2^r$.

Generating function is $G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

Here $a_0 = 2^0, a_1 = 2^1, a_2 = 2^2, \dots$

$\therefore G(x) = 1 + 2x + 2^2 x^2 + 2^3 x^3 + \dots$

$\Rightarrow G(x) = 1 + 2x + (2x)^2 + (2x)^3 + \dots$

$\Rightarrow G(x) = (1-2x)^{-1}$, for $|2x| < 1$, since $(1-x)^{-1} = 1+x+x^2+x^3+\dots, |x|<1$.

$\Rightarrow G(x) = \dfrac{1}{1-2x}$, for $|2x|<1$.

3. Using the generating function, solve the difference equation $y_{n+2} - y_{n+1} - 6y_n = 0, y_1 = 1, y_0 = 2$

Solution:

The given recurrence relation is $y_{n+2} - y_{n+1} - 6y_n = 0, y_1 = 1, y_0 = 2$.

Multiplying the above equation by x^{n+2} and summing from 0 to ∞, we get

$$\sum_{n=0}^{\infty} y_{n+2} x^{n+2} - \sum_{n=0}^{\infty} y_{n+1} x^{n+2} - 6\sum_{n=0}^{\infty} y_n x^{n+2} = 0$$

$$\sum_{n=0}^{\infty} y_{n+2} x^{n+2} - x\sum_{n=0}^{\infty} y_{n+1} x^{n+1} - 6x^2 \sum_{n=0}^{\infty} y_n x^{n} = 0$$

Since $G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$$\therefore \quad G(x) - y_0 - y_1 x - x(G(x) - y_0) - 6x^2 G(x) = 0$$

$$\Rightarrow \quad G(x)(1 - x - 6x^2) - 2 - x + 2x = 0$$

$$\Rightarrow \quad G(x) = \frac{2-x}{1-x-6x^2} = \frac{x-2}{6x^2+x-1} \qquad \dots\dots\dots (1)$$

Using the partial fractions, we have

$$\frac{x-2}{6x^2+x-1}=\frac{x-2}{6x^2+3x-2x-1}=\frac{x-2}{(3x-1)(2x+1)}$$

$$\frac{x-2}{(3x-1)(2x+1)}=\frac{A}{3x-1}+\frac{B}{2x+1}$$

$$x-2=A(2x+1)+B(3x-1)$$

Put $x=-\frac{1}{2}\Rightarrow -\frac{1}{2}-2=B\left(\frac{-3}{2}-1\right)\Rightarrow \frac{-5}{2}=\frac{-5}{2}B\Rightarrow B=1$

Put $x=\frac{1}{3}\Rightarrow \frac{1}{3}-2=A\left(\frac{2}{3}+1\right)\Rightarrow \frac{-5}{3}=\frac{5}{3}A\Rightarrow A=-1$

$$\therefore\ \frac{x-2}{(3x-1)(2x+1)}=\frac{-1}{3x-1}+\frac{1}{2x+1}$$

$$\therefore\ (1)\Rightarrow G(x)=\frac{-1}{3x-1}+\frac{1}{2x+1}$$

$$\Rightarrow G(x)=\frac{1}{1-3x}+\frac{1}{1+2x}=(1-3x)^{-1}+(1+2x)^{-1}$$

$$\Rightarrow G(x)=\left(1+3x+(3x)^2+(3x)^3+\ldots+(3x)^n+\ \ldots\right)+\left(1-2x+(2x)^2-(2x)^3+\ldots+(-1)^n(2x)^n+\ldots\right)$$

$$\Rightarrow \sum_{n=0}^{\infty}a_n y^n=\sum_{n=0}^{\infty}(3x)^n+\sum_{n=0}^{\infty}(-1)^n(2x)^n$$

y_n = Coefficient of x^n in G(x).

$$\therefore\ y_n=3^n+(-2)^n.$$

4. Using method of generating function to solve the recurrence relation, $a_n=4a_{n-1}-4a_{n-2}+4^n; n\geq 2$, given that $a_0=2$ and $a_1=8$.

Solution:

The given recurrence relation is $a_n-4a_{n-1}+4a_{n-2}=4^n; n\geq 2$.

Multiplying the above equation by x^n and summing from 2 to ∞, we get

$$\sum_{n=2}^{\infty}a_n x^n-4\sum_{n=2}^{\infty}a_{n-1}x^n+4\sum_{n=2}^{\infty}a_{n-2}x^n=\sum_{n=2}^{\infty}(4x)^n$$

$$\sum_{n=2}^{\infty} a_n x^n - 4x\sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 4x^2\sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 16x^2(1-4x)^{-1}$$

Since $G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$$\therefore \quad G(x) - a_0 - a_1 x - 4x(G(x) - a_0) + 4x^2 G(x) = \frac{16x^2}{1-4x}$$

$$\Rightarrow \quad G(x)(1-4x+4x^2) - 2 - 8x + 8x = \frac{16x^2}{1-4x}$$

$$\Rightarrow \quad G(x)(1-4x+4x^2) = \frac{16x^2}{1-4x} + 2$$

$$\Rightarrow \quad G(x) = \frac{16x^2 - 8x + 2}{(1-4x)(1-4x+4x^2)} = \frac{16x^2 - 8x + 2}{(1-4x)(1-2x)^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Using the partial fractions, we have

$$\frac{16x^2 - 8x + 2}{(1-4x)(1-2x)^2} = \frac{A}{1-4x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2}$$

$$16x^2 - 8x + 2 = A(1-2x)^2 + B(1-4x)(1-2x) + C(1-4x)$$

Put $x = \frac{1}{2} \Rightarrow 4 - 4 + 2 = C(1-2) \Rightarrow C = -2$

Put $x = \frac{1}{4} \Rightarrow 1 - 2 + 2 = A\left(1 - \frac{1}{2}\right)^2 \Rightarrow 1 = A\left(\frac{1}{4}\right) \Rightarrow A = 4$

Put $x = 0 \Rightarrow 2 = A + B + C \Rightarrow 2 = 4 + B - 2 \Rightarrow B = 0$

$$\therefore \quad \frac{16x^2 - 8x + 2}{(1-4x)(1-2x)^2} = \frac{4}{1-4x} - \frac{2}{(1-2x)^2}$$

$$\therefore \quad (1) \Rightarrow G(x) = \frac{4}{1-4x} - \frac{2}{(1-2x)^2}$$

$$\Rightarrow G(x) = 4(1-4x)^{-1} - 2(1-2x)^{-2}$$

$$\Rightarrow G\ x\ =\ \big(\ +\ x +\ x^2 +\ x^3 +\ + (\ x)^n + 4\ 1\big)$$

$$-2\big(1 + 2(2x) + 3(2x)^2 + 4(2x)^3 + \ldots + (n+1)(2x)^n + \ldots\big)$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = 4\sum_{n=0}^{\infty} (4x)^n - 2\sum_{n=0}^{\infty} (n+1)(2x)^n$$

a_n = Coefficient of x^n in G(x).

$\therefore \quad a_n = 4 \cdot 4^n - 2(n+1)(2)^n$.

5. Find the generating function of Fibonacci sequence.

Solution:

The Fibonacci sequence is 0, 1, 1, 2, 3, 5, 8, 13 …

$\therefore$ The recurrence relation of the Fibonacci sequenceis $a_n = a_{n-1} + a_{n-2}$ with $a_0 = 0, a_1 = 1$.

$\Rightarrow a_n - a_{n-1} - a_{n-2} = 0$.

Multiplying the above equation by x^n and summing from 2 to ∞, we get

$$\sum_{n=2}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\sum_{n=2}^{\infty} a_n x^n - x\sum_{n=2}^{\infty} a_{n-1} x^{n-1} - x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

Since $G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$\therefore \quad G(x) - a_0 - a_1 x - x(G(x) - a_0) - x^2 G(x) = 0$

$\Rightarrow \quad G(x)(1 - x - x^2) - x = 0$

$\Rightarrow \quad G(x) = \dfrac{x}{1 - x - x^2}$.

6. Using generating function solve the recurrence relation, $a_n = 3a_{n-1}$ for $n \geq 1$ with $a_0 = 2$

Solution:

$$Let \qquad G(x) = \sum_{n=0}^{\infty} a_n x^n$$

The given recurrence relation can be written as

$a_n - 3a_{n-1} = 0$

Multiplying the above equation by x

$$\rightarrow \qquad \sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} 3a_{n-1} x^n {}_{==0}$$

$$\rightarrow \quad \sum_{n=1}^{\infty} a_n x^n - 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 0$$

$$\rightarrow (G(x) - a_0) - 3x\, G(x) = 0$$

$$G(x) - 3x\, G(x) = a_0$$

$$G(x)[1 - 3x] = a_0$$

$$G(x) = \frac{a_0}{1-3x} \qquad \therefore \text{given } a_0 = 2$$

$$\therefore\ G(x) = \frac{2}{1-3x} = 2[1-3x]^{-1}$$

$$= 2[1 + 3x + (3x)^2 + (3x)^3 + \ldots\ldots\ldots]^{-1}$$

$$= 2 \sum_{n=0}^{\infty} 3^n . x^n$$

$$\sum_{n=0}^{\infty} a^n x^n = \sum_{n=0}^{\infty} 23^n x^n$$

Equating coefficients of x^n both sides, we get

$$a_n = 23^n$$

Which is required solution.

7. Use method of generating function to solve the recurrence relation $a_n = 3a_{n-1}+1$; $n \geq 1$ given that $a_0 = 1$

Solution:

$$Let \quad G(x) = \sum_{n=0}^{\infty} a_n x^n$$

The given recurrence relation can be written as

$$a_n - 3a_{n-1} = 1$$

Multiplying the above equation by x^n and summing from 1 to ∞ we get

$$\sum_{n=1}^{\infty} a_n x^n - 3 \sum_{n=0}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} (1)\, x^n$$

$$\sum_{n=1}^{\infty} a_n x^n - 3x \sum_{n=0}^{\infty} a_{n-1} x^{n-1} = x + x^2 + x^3 + \cdots$$

$(G(x) - a_0) - 3xG(x) = x\,[1 + x + x^2 + \cdots]$

$G(x)(1 - 3x) = a_0 + \dfrac{x}{1-x}$ $\boxed{\therefore\ a\mathbf{0} = \mathbf{1}}$

$\dfrac{x}{1-x}$

$= \dfrac{1-x+x}{1-x}$

$$\boxed{G(x) = \frac{1}{(1-x)(1-3x)}}$$

By applying partial fraction, we've

$G(x) \quad = \dfrac{A}{(1-x)} + \dfrac{B}{(1-3x)}$

$= \dfrac{A(1-3x)B(1-x)}{(1-x)(1-3x)}$

$1 = A(1 - 3x) + B\,(1 - x)$

put $x = 1/3$, we get

$1 = 0 + B(2/3)$

$\boxed{\mathbf{B = 3/2}}$

put $x = 1$, we get

$1 = 0 + A(1 - 3) + 0$

$\boxed{\mathbf{A = -1/2}}$

Now

$G(x) \quad = \dfrac{-1/2}{1-x} + \dfrac{3/2}{1-3x}$

$= \dfrac{1}{2}[1-x]^{-1} + \dfrac{3}{2}[1-3x]^{-1}$

$= \dfrac{1}{2}[1-x]^{-1} + \dfrac{3}{2}[1-3x]^{-1}$

$= -\dfrac{1}{2}[1+x+x^2+\ .....] + \dfrac{3}{2}[1+(3x)+(3x)^2+........]$

$(is) \qquad \sum_{n=0}^{\alpha} a_n x^n = -\frac{1}{2}\sum_{n=0}^{\alpha} x^n + \frac{3}{2}\sum_{n=0}^{\alpha} 3^n x^n$

Equating efficients of x^n both sides

$$a_n = -\frac{1}{2}(1) + \frac{3}{2}(3)^n$$

$$\text{(is) } a_n = \frac{1}{2}[3^{n+1} - 1]$$

Which is required solution.

8. Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$, n ≥ 2 given that $a_0 = 10$, $a_1 = 41$ using generating functions

Solution:

$$Let \quad G(x) = \sum_{n=0}^{\infty} a_n x^n$$

The given recurrence relation can be written as

$$a_n - 7a_{n-1} + 10\, a_{n-2} = 0$$

Multiplying the above equation by x^n and summing from 2 to ∞ we get

$$\sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1} x^n + 10\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\sum_{n=2}^{\infty} a_n x^n - 7x\sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 10x^2\sum_{n=1}^{\infty} a_{n-2} x^{n-2} = 0$$

$\rightarrow [G(x) - a_0 - a_1 x] - 7x[G(x) - a_0] + 10x^2 G(x) = 0$

$G(x) - a_0 - a_1 x - 7xG(x) + 7xa_0 + 10x^2\, G(x) = 0$

$G(x)\,[1 - 7x10x^2] = a_0 + a_1 x - 7xa_0$ $\boxed{\therefore a\mathbf{0} = \mathbf{10},\ a\mathbf{1} = \mathbf{41}}$

$G(x)[1 - 7x + 10x^2] = 10 + 41x - 70x$

$$G(x) = \frac{10 - 29x}{10x^2 - 7x + 1}$$

$$G(x) = \frac{10 - 29x}{(1 - 2x)(1 - 5x)}$$

$$G(x) = \frac{A}{(1 - 2x)} + \frac{A}{(1 - 5x)}$$

$= A\,[1 - 2x]^{-1} + B\,[1 - 5x]^{-1}$

$= A\,[1 + (2x) + (2x)^2 + \dots] + B[1 + (5x) + (5x)^2 + \dots]$

$$\sum_{n=0}^{\infty} a_n x^n = A \sum_{n=0}^{\infty} 2^n x^n + B \sum_{n=0}^{\infty} 5^n x^n$$

Equating co-eff of x^n both sides

$$a_n = A\,2^n + B\,5^n,\ n \geq 2 \qquad \rightarrow (1)$$

Thus, given $a_0 = 10$, Put n = 0 in (1)

$$10 = A + B \qquad \rightarrow (2)$$

Given $a_1 = 41$, Put n = 1 in (1)

$$41 = 2A + 5B \qquad \rightarrow (3)$$

Solving (2) & (3), we set A = 3, B = 7

Put in (1)

$$a_n = 32^n + 75^n$$

Which is required solution.

9. Use the method of generating function to solve me recurrence relation

$a_n = 4a_{n-1} + 3n.\ 2^n$

Solution:

$$Let \qquad G(x) = \sum_{n=0}^{\alpha} a_n x^n$$

The given recurrence relation equation can be written as

$$a_n = 4a_{n-1} + 3n.\ 2^n$$

Multiply the above equation by x^n and summing from 1 to ∞, we get

$$\sum_{n=1}^{\infty} a_n x^n - 4 \sum_{n=1}^{\infty} a_{n-1} x^n = 3 \sum_{n=1}^{\infty} n 2^n x^n$$

$$\sum_{n=1}^{\infty} a_n x^n - 4x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 3(2)(x) \sum_{n=1}^{\infty} n\,(2x)^{n-1}$$

$[G(x) - a_o] - 4xG(x) = 6x\,[1 + 2(2x) + 3(2x)^2 + \dots]$

$[G(x) -]\ a_o + 4xG(x) = 6x\,[1 - 2x]^{-2}$

$G(x)$ {1-4x}= $\dfrac{6x}{(1-2x)^2} a_0$ // $a_0 = 4$

$$G(x) = \frac{6x}{[1-2x]^2 - [1-4x]} + \frac{6}{[1-4x]}$$

Applying partial fraction, we get

$$G(x) = \frac{A}{(1-2x)} + \frac{B}{(1-2x)^2} + \frac{C}{(1-4x)} + \frac{4}{(1-4x)}$$

Solving we get A = -3, B = -3, C = 10

$$G(x) = \frac{-3}{(1-2x)} - \frac{3}{(1-2x)^2} + \frac{10}{(1-4x)} + \frac{4}{(1-4x)}$$

$= - 3[1 - 2x]^{-1} - 3[1 - 2x]^{-2} + 10[1 - 4x]^{-1} + 4[1 - 4x]^{-1}$

$= -3[1 + 2x + (2x)^2 + \ldots] - 3[1 + 2x + 3(2x)^2 + \ldots] + 10[1 + 4x + (4x)^2 + \ldots]$

$+4[1 + 4x + (4x)^2 + \ldots]$

$$\sum_{n=0}^{\infty} a_n x^n = -3 \sum_{n=0}^{\infty} 2^n x^n - 3 \sum_{n=0}^{\infty} (n+1)\, 2^n x^n + 10 \sum_{n=0}^{\infty} 4^n x^n + 4 \sum_{n=\infty 0}^{\infty} 4^n x^n$$

Equating co. eff of x^n, we get

$a_n = - 3\, 2^n - 3\,(n+1)\, 2^n + 10\, 4^n + 4\, 4^n$

$a_n = 14\, (4)^n - 3\, 2^n - 3\,(n + 1)\,(2)^n$

Which is required solution.

10. Find the sequence having the expression $\dfrac{3-5x}{1-2x-3x^2}$ as a generating function

Solution:

$$Let \quad G(x) = \sum_{n=0}^{\alpha} a_n x^n$$

Given expression is written as

$$G(x) = \frac{3-5x}{(1-2x-3x2)} = \frac{3-5x}{(1-3x)(1+x)}$$

$$= \frac{A}{(1-3x)} + \frac{B}{(1+x)}$$

$3 - 5x = A\,(1 + x) + B\,(1 - 3\,x)$

Put $x = -1$, we get

$3+5 = 0 + B\,(1+3)$

$\boxed{\mathbf{B = 2}}$

Put $x = 1/3$, we get

$3 - 5/3 = A\,(1+1/3) + 0$

$\frac{4}{3} = A\frac{4}{3}$

$\boxed{\mathbf{A = 1}}$

$G(x) \quad = \frac{A}{1-3x} + \frac{B}{1+x}$

$= [1 - 3x]^{-1} + 2[1 - x]^{-1}$

$= [1+(3x) + (3x)^2 + \ldots.] + 2[1 - x + x^2 - x^3 + \ldots]$

$$\sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty} 3^n . x^n + 2\sum_{n=0}^{\infty} (-1)^n . x^n$$

Equating co-eff of x^n, we get

$\boxed{a_n = 3^{n+2}(-1)^n}$

Which is required solution.

11. Using generating function solve the recurrence relation corresponding to the Fibonacci sequence, $a_n = a_{n-1} + a_{n-2}$, $n \geq 2$ with $a_0 = 1$, $a_1 = 1$

Solution:

$$Let \quad G(x) = \sum_{n=0}^{\infty} a_n\, x^n$$

Given recurrence relation is $a_n = a_{n-1} + a_{n-2} = 0$

Multiply the above equation by xn and summing from 2 to ∞, we get

$$\sum_{n=2}^{\infty} a_n\, x^n - \sum_{n=2}^{\infty} a_{n-1}\, x^n - \sum_{n=2}^{\infty} a_{n-2}\, x^{n-2} \quad = 0$$

$$\rightarrow \qquad \sum_{n=2}^{\infty} a_n\, x^n - x\sum_{n=2}^{\infty} a_{n-1}\, x^{n-1} - x^2\sum_{n=2}^{\infty} a_{n-2}\, x^{n-2} \quad = 0$$

[

$G(x)$ - a_0 -$a_1(x)$] - x [$G(x)$ -a_0] - $x2G(x)$ = 0

$G(x) - a_0 - a_1x - x\, G(x) + xa_0 - x^2G(x) = 0$

$G(x)\ \{1 - x - x^2\} = a_0 + a_1x - x\, a_0$

$$\therefore G(x) = \frac{1}{1-x-x^2}$$

$$\therefore G(x) = \frac{1}{\left\{1-\frac{1+\sqrt{5}}{2}x\right\}\left\{1-\frac{1-\sqrt{5}}{2}x\right\}}$$

Applying partial fraction, we get

$$G(x) = \frac{A}{\left\{1-\frac{1+\sqrt{5}}{2}x\right\}} + \frac{B}{\left\{1-\frac{1-\sqrt{5}}{2}x\right\}}$$

Solve, we get $A = \frac{1}{2\sqrt{5}}(1+\sqrt{5}), B = \frac{(1-\sqrt{5})}{-2\sqrt{5}}$

Now

$$G(x) = \frac{\frac{1}{2\sqrt{5}}(1+\sqrt{5})}{\left\{1-\frac{1+\sqrt{5}}{2}x\right\}} + \frac{\frac{1-\sqrt{5}}{2\sqrt{5}}}{\left\{1-\frac{1-\sqrt{5}}{2}x\right\}}$$

$$= \frac{(1+\sqrt{5})}{2\sqrt{5}}\left\{1+\frac{1+\sqrt{5}}{2}x+\left(\frac{1+\sqrt{5}}{2}x\right)^2 + \quad\right\}$$

$$= \frac{(1-\sqrt{5})}{2\sqrt{5}}\left\{1+\frac{1-\sqrt{5}}{2}x+\left(\frac{1+\sqrt{5}}{2}x\right)^2 + \quad\right\}$$

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1+\sqrt{5}}{2\sqrt{5}}\sum_{n=0}^{\infty}\left(\frac{1+\sqrt{5}}{2}\right)^n.x^n - \frac{1-\sqrt{5}}{2\sqrt{5}}\sum_{n=0}^{\infty}\left(\frac{1-\sqrt{5}}{2}\right)^n.x^n$$

Equating co-eff of xn, we get

$$a_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^{n+1} - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^{n+1}$$

Which is required solution.

$$|A \cap B| = \left\lfloor \frac{1000}{7 \times 11} \right\rfloor = [12.9] = 12$$

$|A \cup B|$.

By the principle of inclusion – exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$

$$\therefore |A \cup B| = 142 + 90 - 12 = 220.$$

There are 220 positive integers not exceeding 1000 that are divisible by either 7 or 11.

2. Find the number of integers between 1 and 250 both inclusive that are divisible by any of the integers 2, 3, 5, 7.

Solution:

Let *A*, *B*, *C*, *D* denote the number of integers between 1 and 250 both inclusive that are divisible by 2, 3, 5 and 7 respectively.

$$|A| = \left\lfloor \frac{250}{2} \right\rfloor = [250] = 250$$

$$|B| = \left\lfloor \frac{250}{3} \right\rfloor = [83.33] = 83$$

$$|C| = \left\lfloor \frac{250}{5} \right\rfloor = [50] = 50$$

$$|D| = \left\lfloor \frac{250}{7} \right\rfloor = [35.71] = 35$$

Now, the number of integers between 1 and 250 both inclusive that are divisible by 2 and 3

$$= |A \cap B| = \left\lfloor \frac{250}{2 \times 3} \right\rfloor = [41.66] = 41$$

Similarly, $|A \cap C| = \left\lfloor \frac{250}{2 \times 5} \right\rfloor = [25] = 25$

$$|A \cap D| = \left\lfloor \frac{250}{2 \times 7} \right\rfloor = [17.85] = 17$$

$$|B \cap C| = \left\lfloor \frac{250}{3 \times 5} \right\rfloor = [16.66] = 16$$

$$|B \cap D| = \left\lfloor \frac{250}{3 \times 7} \right\rfloor = [11.90] = 11$$

$$|C \cap D| = \left\lfloor \frac{250}{5\times 7} \right\rfloor = [7.14] = 7$$

Now, the number of integers between 1 and 250 both inclusive that are divisible by 2, 3 and 5

$$= |A \cap B \cap C| = \left\lfloor \frac{250}{2\times 3\times 5} \right\rfloor = [8.33] = 8$$

Similarly, $$|A \cap B \cap D| = \left\lfloor \frac{250}{2\times 3\times 7} \right\rfloor = [5.95] = 5$$

$$|A \cap C \cap D| = \left\lfloor \frac{250}{2\times 5\times 7} \right\rfloor = [3.57] = 3$$

$$|B \cap C \cap D| = \left\lfloor \frac{250}{3\times 5\times 7} \right\rfloor = [2.38] = 2$$

Now, the number of integers between 1 and 250 both inclusive that are divisible by 2, 3, 5 and 7

$$= |A \cap B \cap C \cap D| = \left\lfloor \frac{250}{2\times 3\times 5\times 7} \right\rfloor = [1.19] = 1$$

$$|A \cup B \cup C \cup D|.$$

By the principle of inclusion – exclusion:

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$$

$$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

$$\therefore |A \cup B \cup C \cup D| = 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7 + 8 + 5 + 3 + 2 - 1 = 193$$

There are 193 integers between 1 and 250 both inclusive that are divisible by any of the integers 2,3, 5, 7.

3. Find the number of integers between 1 and 250 that are not divisible by any of the integers 2, 3, 5, 7.

Solution:

By the previous problem, the number of integers between 1 and 250 that are divisible by any of the integers 2,3, 5, 7 is $|A \cup B \cup C \cup D| = 193$.

$\therefore$ The number of integers between 1 and 250 that are not divisible by any of the integers 2, 3, 5, 7

$$= \text{Total} - |A \cup B \cup C \cup D| = 250 - 193 = 57.$$

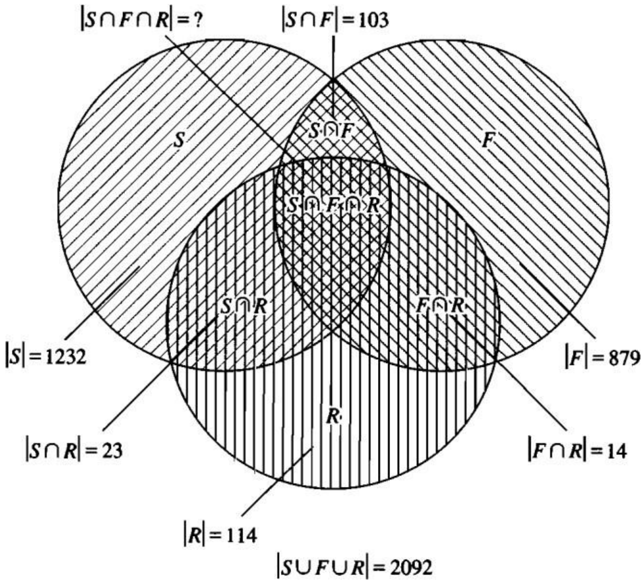
4. A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken atleast one of Spanish, French and Russian, how many students have taken a course in all three languages?

Solution:

Let S be the students who have taken a course in Spanish.

Let F be the students who have taken a course in French.

Let R be the students who have taken a course in Russian.



Then $|S| = 1232$; $|F| = 879$; $|R| = 114$; $|S \cap F| = 103$; $|S \cap R| = 23$; $|F \cap R| = 14$; $|S \cup F \cup R| = 2092$.

By the principle of inclusion – exclusion, we have

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$$

$$\therefore |S \cap F \cap R| = 2092 - 2225 + 140$$

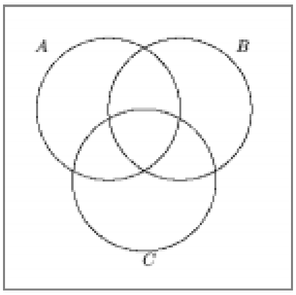
$$\therefore |S \cap F \cap R| = 7.$$

Therefore, there are 7 students who have taken the course in Spanish, French and Russian.

5. In a survey of 100 students, it was found that 30 studied Mathematics, 54 studied Statistics, 25 studied Operation Research, 1 studied all the three subjects. 20 studied Mathematics and Statistics, 3 studied Mathematics and Operations Research and 15 studied Statistics and Operations Research.

   (i) How many students studied none of these subjects?

   (ii) How many students studied only Mathematics?



Solution:

Let A = Students who studied Mathematics

B = Students who studied Statistics

C = Students who studied Operation Research

From the data, we get

$|A| = 30$, $|B| = 54$, - $|C| = 25$, $|A \cap B| = 20$, $|A \cap C| = 3$, $|B \cap C| = 15$

$|A \cap B \cap C| =1$. Total students = 100

(i) Student who studied any one subject is

$|A \cup B \cup C| \quad = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 30+54+25-20-15-3+1$

$|A \cup B \cup C| \quad = 72.$

$\therefore$ Student who studied none of 3 subjects = 100 - 72

= 28 Students

(ii) The number of students studied only Mathematics and Statistics

$= |A \cap B| - |A \cap B \cap C|$

$= 20 - 1 = 19$

The number of students studied only Mathematics and Operation Research

$= 3-1 = 2$

$\therefore$ The number of students studied only mathematics is

$= 30-19 = 2$

= 9 students.

6. There are 2500 students in a college, of these 1700 have taken a course in C, 1000 have taken a course Pascal and 550 have taken a course in Networking. Further 750 have taken courses in both C and Pascal, 400 have taken courses in both C and Networking, and 275 have taken courses in both Pascal and Networking. If 200 of these students have taken courses in C, Pascal and Networking.

   (i) How many of these 2500 students have taken a course in any of these three courses C, Pascal and Networking?

   (ii) How many of these 2500 students have not taken a course in any of these three courses C, Pascal and Networking?

Solution:

Let A, B and C denotes Students have taken a course C, Pascal and Networking respectively.

From the given data, we get

$|A| = 1700$, $|B| = 1000$, - $|C| = 550$, $|A \cap B| = 750$, $|A \cap C| = 400$, $|B \cap C| = 275$,

$|A \cup B \cup C| = 200$

By principle of Inclusion - Exclusion we've

(i) Any one of these three courses

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= (1700+1000+550) - (750+400+275) + 200$

$= 3450 - 1425$

$\boxed{\therefore |A \cup B \cup C| = 2025}$

(ii) Number of students who have not taken any 3 courses

$=$ Total- $|A \cup B \cup C|$

$= 2500 - 2025$

$= 475$

UNIT-4

LATTICES AND BOOLEAN ALGEBRA

Introduction

Boolean algebras are key structures in mathematics that model logical operations. They consist of sets with operations like meet, join, and complement, following specific rules that reflect logical reasoning, making them essential in both lattice and order theory. Boolean algebras are foundational in computer science, particularly in digital logic design and circuit analysis. They are also essential in set theory, providing a framework for understanding unions, intersections, and complements of sets.

4.0 Relation:

A relation R on a set A is a well defined rule, which tells whether given two elements x and y of A are related or not. If x is related to y, we write x R y, otherwise x R y.

Note:

If A is a finite set with "n" elements then A ×A has n^2 elements. Therefore, A×A has 2^{n^2} relation on an "n" element set.

Reflexive:

Let X be a set and R be a relation defined on X. Then R is said to be reflexive if it satisfies the following condition

$x \, R \, x$, for all $x \in X$.

Symmetric:

Let X be a set and R be a relation defined on X. Then R is said to be symmetric if it satisfies the following condition

$x \, R \, y \Rightarrow y \, R \, x$, for all $x, y \in X$.

Note:

A relation which is not symmetric is called asymmetric.

Antisymmetric:

Let X be a set and R be a relation defined on X. Then R is said to be antisymmetric if it satisfies the following condition

$$x \, R \, y \ \& \ y \, R \, x \Rightarrow x = y, \quad \text{for all } x, y \in X.$$

Transitive:

Let X be a set and R be a relation defined on X. Then R is said to be transitive if it satisfies the following condition

x R y&y R $z \Rightarrow x$ R z, for all $x, y, z \in$ X.

Equivalence relation:

A relation R on a set A is called an equivalence relation, if R is reflexive, symmetric and transitive.

Example:

Let A = {a, b, c} be a set and R = {(a, a), (b, b), (c, c), (a, b), (b, a)} be a relation defined on A. Then R is an equivalence relation.

4.1 Partially ordered set (Poset):

A relation R on a set A is called a partial order relation, if R is reflexive, antisymmetric and transitive. The set A together with partial order relation R is called partially ordered set or poset.

Example:

Divides relation (/) is a partial order relation.

Problems:

1. Show that (N, $\leq$) is a partially ordered set where N is set of all positive integers and $\leq$ is defined by m $\leq$ n iff n – m is a non-negative integer.

Solution:

Given N is the set of all positive integers.

The given relation is m $\leq$ n if and only if n – m is a non-negative integer.

Reflexive:

For all x $\in$ N, x – x = 0 is non-negative integer.

$\therefore$ x R x, $\forall$ x $\in$ N.

$\therefore$ R is reflexive.

Antisymmetric:

Consider x R y and y R x.

Since x R y $\Rightarrow$ x – y is non-negative integer ....(1)

y R x $\Rightarrow$ y – x is non-negative integer.

$\Rightarrow$– (x – y) is non-negative integer ....(2)

From (1) and (2), we have

$$x = y.$$

Therefore R is antisymmetric.

Transitive:

Assume that x R y and y R z.

Since x R y $\Rightarrow$ x – y is non-negative integer and

y R z $\Rightarrow$ y – z is non-negative integer.

$\Rightarrow$(x – y) + (y – z) is non-negative integer.

$\Rightarrow$x – z is non-negative integer.

$\Rightarrow$ x R z.

$\therefore$ x R y and y R z $\Rightarrow$ x R z.

$\therefore$ R is transitive.

$\therefore$ R is a partial order relation.

2. Let N be set of all natural numbers. Prove that the relation R in N defined by a Rb if and only if a divides b is a partial order relation.
3. Let R be the set of all real numbers. Prove that the relation S defined by $a \leq b$ than or equal to b) is a partial order relation in R.

Comparable property:

In a poset for any two elements a, b either $a \leq b$ or $b \leq a$ is called comparable property. Otherwise it is incomparable property.

Totally ordered set (Linearly ordered set or chain):

A partially ordered set is said to be totally ordered set, if any two elements are comparable.

Example:

a R b if $a \leq b$ is a total order.

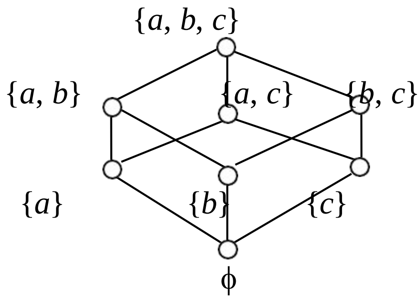
a R b if a / b is a total order.

For, given elements 2 and 3, neither 2 / 3 nor 3 / 2. (i.e., 2 and 3 are not comparable).

Hasse diagram:

Pictorial representation of a poset is called Hasse diagram.

Example:



This Hasse diagram can be represented as $(\rho(A), \subseteq)$, where $A = \{a, b, c\}$.

Example:

The greater than or equal to ($\geq$) relation is a partial ordering on the set of integers Z.

Problems:

1. Draw Hasse diagram for {(a, b) | a divides b} on X = {2, 3, 6, 12, 24, 36}.

Solution:

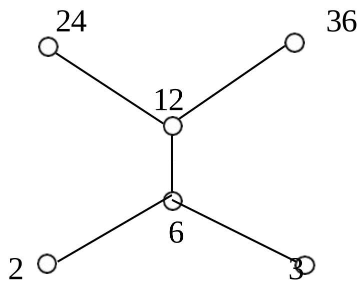
The relation R = {(2,6), (2, 12), (2, 24), (2, 36),

(3,6), (3, 12), (3, 24), (3, 36),

(6, 12), (6, 24), (6, 36),

(12, 24), (12, 36)}.

The Hasse diagram for (X, R) is



2. Draw the Hasse diagram of $\langle X, \leq \rangle$, where $X = \{2, 4, 5, 10, 12, 20, 25\}$ and the relation $\leq$ be such that $x \leq y$ is x and y.

Solution:

The relation R = {(2, 4), (2, 5), (2, 10), (2, 12),(2, 20),(2, 25),

(4, 5), (4, 10), (4, 12),(4, 20),(4, 25),

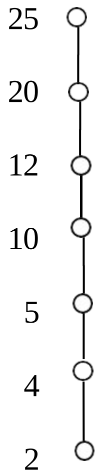
(5, 10), (5, 12),(5, 20),(5, 25),

(10, 12),(10, 20),(10, 25),

(12, 20),(12, 25),

(20, 25)}.

The Hasse diagram for (X, ≤) is



Greatest element and least element:

An element $a \in A$ is called the greatest element of A, if $a \geq x, \forall x \in A$.

An element $a \in A$ is called a least element of A if $a \leq x, \forall x \in A$.

In the poset $(P(A), \subseteq)$ the least element is ϕ and the greatest element is A.

We can observe from the Hasse diagram, the least element is at the lower level and the greatest element is at higher level.

Upper bound and lower bound of a set:

Consider a poset $(A, \leq)$ and a subset B of A. The element $a \in A$ is called an upper bound of B if $b \leq a$ for every $b \in B$.

An element "a" is called a lower bound of B if $a \leq b, \forall b \in B$.

Least upper bound (LUB) (or) Supremum:

An element $a \in A$ is called least upper bound of B if "a" is an upper bound of B and $a \leq a'$ whenever "a" is an upper bound of B.

Greatest lower bound (GLB) (or) Infimum:

An element $a \in A$ is called a greatest lower bound of B if "a" is a lower bound of B and $a' \leq a$ whenever "a" is a lower bound of B.

General formula:

i) GLB$\{a, b\} = a*b = a\wedge b.$

ii) LUB$\{a, b\} = a\oplus b = a\vee b.$

iii) $a\wedge b \leq a$ & $a\wedge b \leq b$ $a\vee b \geq a$ & $a\vee b \geq b$

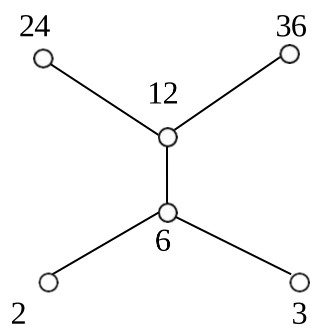
iv) If $a \leq b$, then $a\wedge b = a$ and $a\vee b = b$

Example:

Let X = {2, 3, 6, 12, 24, 36} and the relation $\leq$ be such that $x \leq y$ if and only if x divides y.

Clearly $(X, \leq)$ is a poset.

The Hasse diagram of $(X, \leq)$ is given by



UB {2, 3} = {6, 12, 24, 36}.

LUB (2, 3} = 6.

UB (24, 36} = does not exist.

LUB {24, 36} = does not exist.

LB {2, 3} = does not exist.

GLB {2, 3} = does not exist.

LB {24, 36} = {2, 3, 6, 12}.

GLB {24, 36} = 2.

Problems:

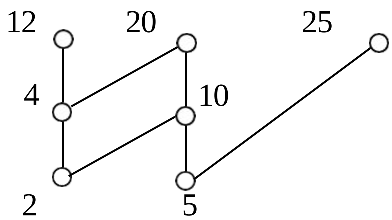
1. Which elements of the poset ({2, 4, 5, 10, 12, 20, 25}, ∕) are maximal and which are minimal?

Solution:

The relation R is R = {(2, 4),(2, 10),(2, 12),(2, 20),(4, 12),(4, 20),

(5, 10),(5, 20),(5, 25),(10, 20)}

It‟s Hasse diagram is



The maximal elements are 12, 20 and 25.

The minimal elements are 2 and 5.

2. Find the greatest lower bound and the least upper bound of the sets {3, 9, 12} and {1, 2, 4, 5, 10}, if they exist in the poset (Z^+, ∕).

Solution:

(i) An integer is a lower bound of {3, 9, 12}, if 3, 9 and 12 are divisible by this integer.

Clearly 1 and 3 are two integers divide 3, 9 and 12.

Since 1 ∕ 3, therefore 3 is the greatest lower bound of {3, 9, 12}.

∴ GLB{3, 9, 12} = 3.

An integer is an upper bound for {3, 9, 12} if and only if it is divisible by 3, 9 and 12.

Such integer is least common multiple of 3, 9 and 12, which is 36.

Hence 36 is the least upper bound of {3, 9, 12}.

∴ LUB{3, 9, 12} = 36.

(ii) The only integer which divides {1, 2, 4, 5, 10} is 1.

Hence 1 is the greatest lower bound of {1, 2, 4, 5, 10}.

∴ GLB{1, 2, 4, 5, 10}= 1.

An integer is an upper bound for {1, 2, 4, 5, 10} if and only if it is divisible by 1, 2, 4, 5 and 10.

Such integer is least common multiple of 1, 2, 4, 5 and 10, which is 20.

Hence 20 is the least upper bound of {1, 2, 4, 5, 10}.

∴ LUB{1, 2, 4, 5, 10} = 20.

3. Show that least upper bound of a subset Bin a poset $\langle A, \leq \rangle$is unique is it exists.

Solution:

Given $\langle A, \leq \rangle$ is a poset and $B \subseteq A$.

Let $a, b \in A$ be two least upper bounds of B.

By the definition of least upper bound,

$a \leq c$, where c is any other upper bound of B. (Taking a is a least upper bound)……… (1)

$b \leq c$, where c is any other upper bound of B. (Taking b is a least upper bound).............(2)

Case (i): If $a<b$, then (2) is not true.

Case (ii): If $b<a$, then (1) is not true.

Therefore $a = b$.

Hence least upper bound is unique.

4. If R is the relation on the set of integers such mat(a,b) ∈ R if and only if b = for some positive integer m, show mat R is a partial ordering.

Solution:

Since $a = a^1$, we have $(a, a) \in R$

∴ R is reflexive → **(A)**

Let (a,b) ∈ R and (b,a) ∈ R

by given b = and a = Now a

= () = ()^n

a =a^{mn}

Which means mn= 1 or a = 1 or a = -1

Case-1: If mn = 1, then m = 1, and n = 1

∴ **a = b**

Case-2: If a =1 , then form given

b = 1^m= 1= a

If b=1 , then form given

a = 1^n= 1=b

In either way **a = b**

Case-3: If a =-1 , then b = -1

∴ **a = b**

From the 3 cases, **a = b** Therefore **R is Antisymmetric** → (B)

Let (a,b) ∈ R and (b,c) ∈ R

(ie) b = a^m and c = b^n

∴ c = = ()^n =

→ c = ()^n →

→ c = → (a,c) ∈ R ∵ **R is transitive** → (C)

Thus R is a partial order relation.

5. Consider x = {1,2,3,4,6,12}, R = {<a, b> / (a/b)}. Find L∪B and GLB for the poset (X, R)

Solution:

The Relation is

R = { <1,2>, <1,3>, <1,4>, <1,6>, <1,12>, <2,4>, <2,6>, <2,12>, <3,6>, <3,12>, <4,12>, <6,12> }

The Hasse diagram is

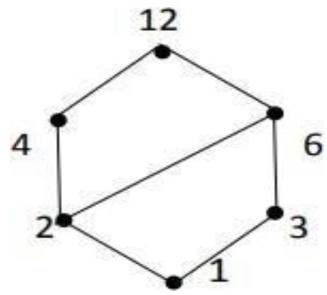
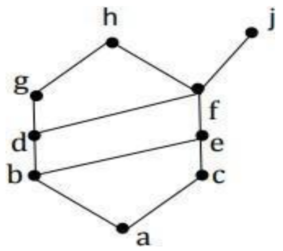


Table of LUB & GLB

S.No	LUB	S.No	GLB
1.	UB {1,3} = {3,6,12} LUB = {3}	1.	LB {1,3} = {1} GLB {1,3} = {1}
2.	UB {1,2,3} = {6,12} LUB = {6}	2.	LB {1,2,3} = {1} GLB {1,2,3} = {1}
3.	UB {2,3} = {6,12} LUB = {6}	3.	LB {2,3} = {1} GLB {2,3} = {1}
4.	UB {2,3,6} = {6,12} LUB = {6}	4.	LB {2,3,6} = {1} GLB {2,3,6} = {1}
5.	UB {4,6} = {12} LUB = {4,6} = {12}	5.	LB {4,6} = {1,2} GLB {4,6} = {2}

6. Find the GLB and LUB of {b, d, g} if they exists in the poset given below.



Solution:

The upper bound of {b, d, g} are given g and h since g<h, g is the least upper bound, **LUB{b,d,g} = g**

The lower bound of {b, d, g} = {b,a}

Since b>a, b is the greatest lower bound

∴ GLB {b,d,g} = b

4.2 Lattices as Posets:

A lattice is a partially ordered set $(L, \leq)$ in which every pair of elements $a, b \in L$ has a greatest lower bound (GLB) and least upper bound (LUB) exists.

Example:

Let A be any finite set. Then $(\rho(A), \subseteq)$ is a lattice.

4.3 Properties of lattice:

Idempotent law	$a \wedge a = a$	$a \vee a = a$
Commutative law	$a \wedge b = b \wedge a$	$a \vee b = b \vee a$
Associative law	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$	$a \vee (b \vee c) = (a \vee b) \vee c$
Absorption law	$a \wedge (a \vee b) = a$	$a \vee (a \wedge b) = a$

Proof:

Idempotent law:

$a \vee a = \text{LUB}(a, a) = \text{LUB}(a) = a$

$a \wedge a = \text{GLB}(a, a) = \text{GLB}(a) = a$

$\therefore a \wedge a = a$ and $a \vee a = a$.

Commutative law:

$a \vee b = \text{LUB}(a, b) = \text{LUB}(b, a) = b \vee a$

$a \wedge b = \text{GLB}(a, b) = \text{GLB}(b, a) = b \wedge a$

$\therefore a \wedge b = b \wedge a$ and $a \vee b = b \vee a$.

Associative law:

Let $a \vee (b \vee c) = d$ .... (1)

and $(a \vee b) \vee c = e$ .... (2)

(1) $\Rightarrow d$ is LUB of $(a, b \vee c)$

$\Rightarrow d \geq a$ and $d \geq b \vee c$ .... (3)

We know that $b \vee c$ is LUB (b, c).

$\Rightarrow b \vee c \geq b$ and $b \vee c \geq c$ .... (4)

From (3) and (4), we have

$d \geq a$, $d \geq b$ and $d \geq c$ .... (5)

From (5), d is an UB of (a, b) and $d \geq c$.

$\Rightarrow d \geq a \vee b$ and $d \geq c$.

$\Rightarrow d$ is an UB of $(a \vee b, c)$ .... (6)

From (2), e is LUB of $(a \vee b, c)$..(7)

From (6) and (7), $d \geq e$.

Similarly, we can easily prove $e \geq d$.

$\therefore d = e$.

Using (1) and (2), we have

$a \vee (b \vee c) = (a \vee b) \vee c$

Similarly, we can easily prove $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

Absorption law:

Since $a \wedge b$ is the GLB of $\{a, b\}$, we have

$a \wedge b \leq a$ ....(1)

Obviously $a \leq a$ ....(2)

From (1) and (2), we have

$a \vee (a \wedge b) \leq a$ ....(3)

By the definition of LUB, we have

$a \leq a \vee (a \wedge b)$..(4)

From (3) and (4), we have

$a \vee (a \wedge b) = a$.

Similarly, we can prove $a \wedge (a \vee b) = a$.

Other properties:

Distributive law	$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$	$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
Complement law	$a \wedge a' = 0$	$a \vee a' = 1$
Demorgan's law	$(a \wedge b)' = a' \vee b'$	$(a \vee b)' = a' \wedge b'$

Theorem:

Let (L, $\leq$) be a lattice. Then for every a and b in L,

(a) $a \vee b = b \Leftrightarrow a \leq b \, (or) \; a \vee b = b \Leftrightarrow a \leq b$ and

(b) $a \wedge b = a \Leftrightarrow a \leq b \, (or) \; a \wedge b = a \Leftrightarrow a \leq b$.

Proof:

(a) Suppose that $a \vee b = b$.

Since $a \leq a \vee b \Rightarrow a \leq b$.

Conversely, assume that $a \leq b$.

As, $b \leq b$, from the definition of $a \vee b$, we have

$\therefore a \vee b \leq b$.. (1)

Since $a \vee b$ is an LUB $\{a, b\}$, we have

$b \leq a \vee b$.. (2)

From (1) and (2), we get

$\Rightarrow a \vee b = b$.

$\therefore a \vee b = b$ if and only if $a \leq b$.

(b) Suppose that $a \wedge b = a$.

Since $a \wedge b \leq b \Rightarrow a \leq b$

Conversely, assume that $a \leq b$.

As, $a \leq a$, from the definition of $a \wedge b$, we have

$\therefore a \leq a \wedge b$.. (3)

Since a $\wedge$ b is a GLB$\{a, b\}$, we have

$a \wedge b \leq a$.. (4)

From (3) and (4), $a \wedge b = a$

$\therefore a \wedge b = a$ if and only if $a \leq b$.

Theorem: (Consistency Laws)

Let (L, $\leq$) be a Lattice. Then for $a, b \in$ L, $a \leq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$.

Proof:

First we prove $a \leq b \Leftrightarrow a \wedge b = a$.

Assume that $a \leq b$.

As, $a \leq a$, from the definition of $a \wedge b$, we have

$\therefore a \leq a \wedge b$.. (1)

Since a $\wedge$ b is a GLB$\{a, b\}$, we have

$a \wedge b \leq a$.. (2)

From (1) and (2), $a \wedge b = a$

$\therefore a \wedge b = a$ if and only if $a \leq b$.

Conversely, assume $a \wedge b = a$.

This is possible only when $a \leq b$.

$$\therefore a \wedge b = a \Rightarrow a \leq b.$$

Hence $\quad a \leq b \Leftrightarrow a \wedge b = a$.

Nextwe prove $a \wedge b = a \Leftrightarrow a \vee b = b$.

Suppose that $a \wedge b = a$, then

$b = b \vee (a \wedge b)$ (Absorption Law)

$= b \vee a.$

$\Rightarrow a \vee b = b.$ (Commutative Law)

Conversely, assume $a \vee b = b$.

$a = a \wedge (a \vee b)$ (Absorption Law)

$= a \wedge b$

$a \wedge b = a.$ (Commutative Law)

$\therefore a \wedge b = a \Leftrightarrow a \vee b = b$

Hence $a \leq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$.

Theorem:

Isotonicity Property:

Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$

$$b \leq c \Rightarrow \begin{cases} a \wedge b \leq a \wedge c \\ a \vee b \leq a \vee c \end{cases} \quad (or)\ b \leq c \Rightarrow \begin{cases} a * b \leq a * c \\ \\ a \oplus b \leq a \oplus c \end{cases}.$$

(Associative law)

Proof:

(Commutative law)

Assume that $b \leq c$.

(Associative law)

We know that $b \leq c \Leftrightarrow b \wedge c = b$.

($a \wedge b = a \Leftrightarrow a \leq b$ & $a \wedge a = a$)

Consider, $(a \wedge b) \wedge (a \wedge c)$

$= a \wedge (b \wedge a) \wedge c$

$= a \wedge (a \wedge b) \wedge c$

$= (a \wedge a) \wedge (b \wedge c)$

$= a \wedge b.$

$(a \wedge b) \wedge (a \wedge c) = a \wedge b$.

Hence $b \leq c \Leftrightarrow a \wedge b \leq a \wedge c$.

We know that $b \leq c \Leftrightarrow b \vee c = c$

Consider, $(a \vee b) \vee (a \vee c)$

$= a \vee (b \vee a) \vee c$ (Associative law)

$= a \vee (a \vee b) \vee c$ (Commutative law)

$= (a \vee a) \vee (b \vee c)$ (Associative law)

$= a \vee c.$ ($a \vee b = b \Leftrightarrow a \leq b$ & $a \vee a = a$)

$(a \vee b) \vee (a \vee c) = a \vee c$.

Hence $b \leq c \Leftrightarrow a \vee b \leq a \vee c$.

Distributive inequality of lattice:

Let$(L, \wedge, \vee)$ be a given lattice. Then for any $a, b, c \in L$, the following inequality holds.

i) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ and

ii) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$.

Theorem:

Let $(L, \wedge, \vee)$ be a lattice. Then for all $a, b, c, d \in L$, we have

(i) $a \leq b \Rightarrow a \vee c \leq b \vee c$,

(ii) $a \leq b \Rightarrow a \wedge c \leq b \wedge c$,

(iii) $a \leq b$ and $c \leq d \Rightarrow (a \vee c) \leq (b \vee d)$ and

(iv) $a \leq b$ and $c \leq d \Rightarrow (a \wedge c) \leq (b \wedge d)$.

Proof:

(i) Assume that $a \le b$.

Then by theorem, $a \wedge b = a$ and $a \vee b = b$ ....(1)

Consider, $(a \vee c) \vee (b \vee c) = a \vee (c \vee b) \vee c$ (Associative law)

$= a \vee (b \vee c) \vee c$ (Commutative law) (Associative law)

$= (a \vee b) \vee (c \vee c)$

$= (a \vee b) \vee c$ (Idempotent law)

$= b \vee c$ (Using (1))

$\therefore \quad a \vee c \le b \vee c$

(ii) Consider, $(a \wedge c) \wedge (b \wedge c) = a \wedge (c \wedge b) \wedge c$ (Associative law)

$= a \wedge (b \wedge c) \wedge c$ (Commutative law)

$= (a \wedge b) \wedge (c \wedge c)$ (Associative law)

$= (a \wedge b) \wedge c$ (Idempotent law)

$= b \wedge c$ (Using (1))

$\therefore \quad a \wedge c \le b \wedge c$

(iii) Assume that $a \le b$ and $c \le d$. ………………….. (2)

Then by theorem, $a \vee b = b$ and $c \vee d = d$

Consider, $(a \vee c) \vee (b \vee d) = a \vee (c \vee b) \vee d$ (Associative law)

$= a \vee (b \vee c) \vee d$ (Commutative law)

$= (a \vee b) \vee (c \vee d)$ (Associative law)

$= b \vee d$ (Using (2))

$\therefore \quad a \vee c \le b \vee d$

(iv) Consider, $(a \wedge c) \wedge (b \wedge d) = a \wedge (c \wedge b) \wedge d$ (Associative law)

$= a \wedge (b \wedge c) \wedge d$ (Commutative law)

$= (a \wedge b) \wedge (c \wedge d)$ (Associative law)

$= a \wedge c$ (Using (2))

$\therefore \quad a \wedge c \le b \wedge d$

Theorem:

Let $(L, \wedge, \vee)$ be a distributive lattice. Then $a \vee b = a \vee c$ and $a \wedge b = a \wedge c \Rightarrow b = c \ \forall \ a, b, c \in L$ (This is called cancellation property)

Proof:

Given $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$, $\forall \ a, b, c \in L$ ……………….. (1)

By distributive lattice, $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$.

Therefore $(a \wedge b) \vee c = (a \wedge c) \vee c$ (Using (1))

$= c$ (Absorption law) .... (2)

Now, $(a \vee c) \wedge (b \vee c) = (a \vee b) \wedge (b \vee c)$ (Using (1))

$= (b \vee a) \wedge (b \vee c)$ (Commutative law)

$= b \vee (a \wedge c)$ (Distributive law)

$= b \vee (a \wedge b)$ (Using (1))

$= b \vee (b \wedge a)$ (Commutative law)

$= b$ (Absorption law) .... (3)

From (2) and (3), we have

$$b = c.$$

Hence $a \vee b = a \vee c$ and $a \wedge b = a \wedge c \Rightarrow b = c \ \forall \ a, b, c \in L$.

Distributive lattice:

A lattice $(L, \wedge, \vee)$ is said to be distributive lattice, if $\wedge$ and $\vee$ satisfies the following conditions, $\forall a, b, c \in L$,

D_1: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and

D_2: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Example:

$(\rho(A), \vee, \wedge)$ is a distributive lattice.

Problems:

2. Prove that every chain is a distributive lattice.
3. In a distributive lattice, prove that, if $a \wedge b \leq x \leq a \vee b$, then $x = (a \wedge x) \vee (b \wedge x) \vee (a \wedge b)$.

Modular lattice:

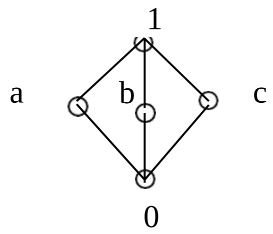
A lattice $(L, \wedge, \vee)$ is said to be modular if for any $a, b, c \in L$,

i. $a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$

ii. $a \geq c \Leftrightarrow a \wedge (b \vee c) = (a \wedge b) \vee c$.

Example:

The diamond lattice or M_5 lattice is a modular lattice.



Theorem:

Prove that every distributive lattice is modular.

Proof:

Let $(L, \leq)$ be a distributive lattice.

Let a, b, c $\in$L such that $a \leq c$.

To prove: $a \vee (b \wedge c) = (a \vee b) \wedge c$

$a \leq c \Rightarrow a \vee c = c$

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$= (a \vee b) \wedge c$

$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$

Therefore $(L, \leq)$ is a modular lattice.

Note:

- Every chain is modular.

  (Since every chain is a distributive lattice and every distributive lattice is modular Every chain is a modular)

- Converse of the theorem is not true.

  Every modular lattice is not distributive.

Problem:

1. If L is a distributive lattice. Prove that

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$$

Solution:

R.H.S $= (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

$= \{a \wedge [(b \vee c) \wedge (c \vee a)]\} \vee [b \wedge [(b \vee c) \wedge (c \vee a)]]$ (Distributive law)

$= \lceil \lfloor \{a \wedge (c \vee a)\} \wedge (b \vee c) \rceil \rfloor \vee \lceil \lfloor \{b \wedge (b \vee c\} \wedge (c \vee a) \rceil \rfloor$ (Associative law)

$= [a \wedge (b \vee c)] \vee [b \wedge (c \vee a)]$ (Absorption law)

$= (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \vee (b \wedge a)$ (Distributive law)

$= (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$. (Commutative and idempotent law)

Therefore $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$.

4.4 Lattices as algebraic systems

Definition

A lattice $(L, \wedge, \vee)$ is said to be algebraic if it satisfies commutative law, associative law, absorption law and existence of idempotent element.

4.5 Sublattice:

Definition

Let $(L, \wedge, \vee)$ be a lattice and let $S \subseteq L$ be a subset of L. Then $(S, \wedge, \vee)$) is a sublattice of $(L, \wedge, \vee)$ if and only if S is closed under both operation $\wedge$ and $\vee$.

$\forall a, b \in S \Rightarrow a \wedge b \in S$ and $a \vee b \in S$.

Example:

(E_+, D) is a sublattice of (Z_+, D), where E_+ is a set of positive even integers and Z_+ is a set of positive integers.

4.6 Direct product and homomorphism

Direct product of lattice:

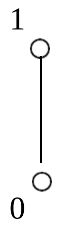
Let $(L, *, \oplus)$ and $(S, \wedge, \vee)$ be two lattices. The algebraic system $(L \times S, \cdot , +)$ in which the binary operation $+$ and $\cdot$ on $L \times S$ are such that for any $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$ in $L \times S$

$$\langle a_1, b_1\rangle \cdot \langle a_2, b_2\rangle = \langle a_1 * a_2, b_1 \wedge b_2\rangle$$

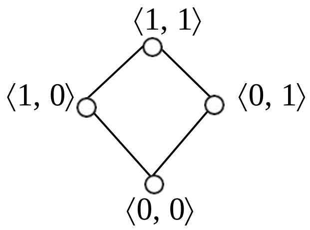
$$\langle a_1, b_1\rangle + \langle a_2, b_2\rangle = \langle a_1 \oplus a_2, b_1 \vee b_2\rangle$$

is called the direct product of the lattice $(L, *, \oplus)$ and $(S, \wedge, \vee)$.

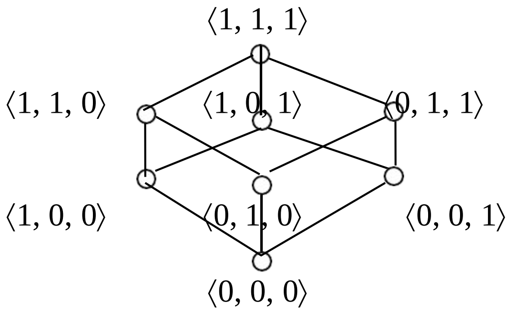
Example:



L



S



L* ×*S

Problem:

1. Show that the direct product of any two distributive lattices is a distributive lattice.

Solution:

Let L_1 and L_2 be two distributive lattices.

Let $x, y, z \in L_1 \times L_2$.(Direct product of L_1 and L_2)

Then $x = (a_1, a_2)$, $y = (b_1, b_2)$ and $z = (c_1, c_2)$ for some $a_1, b_1, c_1 \in L_1$ and $a_2, b_2, c_2 \in L_2$

Now $x \vee (y \wedge z) = (a_1,a_2) \vee (b_1,b_2) \wedge (c_1,c_2)$

$$= (a_1 \vee (b_1 \wedge c_1),\ a_2 \vee (b_2 \wedge c_2))$$

$$= ((a_1 \vee b_1) \wedge (a_1 \wedge c_1),\ (a_2 \vee b_2) \wedge (a_2 \wedge c_2))$$

(Since L_1 and L_2 are distributive)

$$= ((a_1 \vee b_1),(a_2 \vee b_2)) \wedge ((a_1 \vee c_1),(a_2 \vee c_2))$$

$$= ((a_1,a_2) \vee (b_1,b_2)) \wedge ((a_1,a_2) \vee (c_1,c_2))$$

$$= (x \vee y) \wedge (x \vee z)$$

For all $x, y, z \in L_1 \times L_2$, $x \vee (y \wedge z) = (x \vee z) \wedge (x \vee z)$.

Thus if L_1& $L_2$are distributive, then $L_1 \times L_2$is also distributive.

Lattice Homomorphism:

Let $(L_1, \wedge, \vee)$ and $(S, \wedge, \vee)$ be two lattices.

A mapping g: $L_1 \rightarrow L_2$ is called lattice homomorphism if $\forall$ a,b $\in L_1$.

1. $f(a \wedge b) = f(a) * f(b)$ and

2. $f(a \vee b) = f(a) \oplus f(b)$.

A homomorphism which is also 1-1 is called an isomorphism.

4.7 Some Special Lattices

Bounded Lattice:

A lattice $(L, *, \oplus)$is called a bounded lattice, if it has both greatest element 1 and least element 0. It is denoted by $(L, *, \oplus, 0, 1)$.

Example:

$(\rho(A), \subseteq)$ is a bounded lattice.

Here "0" element is ϕ and "1" element is A.

Complemented Lattice:

A bounded lattice $(L, *, \oplus, 0, 1)$is said to be complemented lattice if every element of L has atleast one complement.

Example:

$(\rho(A), \subseteq)$ is a complemented lattice.

Note:

An element may have no complement or may have more than 1 complement.

Complete Lattice:

A lattice $(L, *, \oplus)$ is complete if for all non-empty subsets of L has both GLB and LUB.

Example:

$(\rho(A), \subseteq)$ is a complete lattice.

Problems

1. Show that in a distributive lattice, if complement of an element exits then it must be unique.

Solution:

Let us assume that x and y are two complements of a.

Then $a \wedge x = 0$, $a \vee x = 1$..(1)

$a \wedge y = 0$, $a \vee y = 1$...(2)

Now, $x = x \vee 0$ (Since $x \vee 0 = x$)

$= x \vee (a \wedge y)$ (Using (2))

$= (x \vee a) \wedge (x \vee y)$ (Distributive law)

$= (a \vee x) \wedge (x \vee y)$ (Commutative law)

$= 1 \wedge (x \vee y)$ (Using (1))

$x = x \vee y$...(3)

Now, $y = y \vee 0$ (Since $y \vee 0 = y$)

$= y \vee (a \wedge x)$ (Using (1))

$= (y \vee a) \wedge (y \vee x)$ (Distributive law)

$= (a \vee y) \wedge (y \vee x)$ (Commutative law)

$= 1 \wedge (y \vee x)$ (Using (2))

$y = y \vee x$..(4)

From (3) and (4), we have

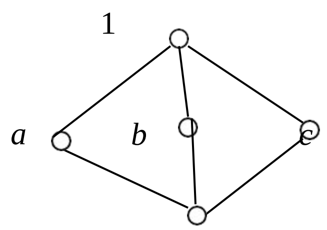
$x = x \vee y = y \vee x = y$.

Therefore complement is unique.

2. Give an example of a lattice but not distributive.

Solution:

Consider the following diamond lattice (or) M_5 lattice.



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Diamond lattice (or) M_5 lattice

For $a \vee (b \wedge c) = a \vee 0 = a$. (Since $b \wedge c = 0$)..(1)

$(a \vee b) \wedge (a \vee c) = 1 \wedge 1 = 1$. (Since $a \vee b = 1$ and $a \vee c = 1$)...............................(2)

Since $a \neq 1$.

$\therefore$ From (1) and (2), we have $a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$.

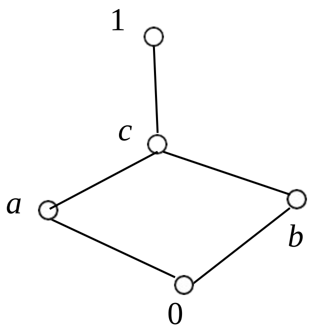
Therefore D_1 is not satisfied.

$\therefore$ The given lattice is not distributive.

3. Give an example of a distributive lattice but not complemented.

Solution:

Consider the following distributive lattice.



Complement of a does not exist.

Complement of b does not exist.

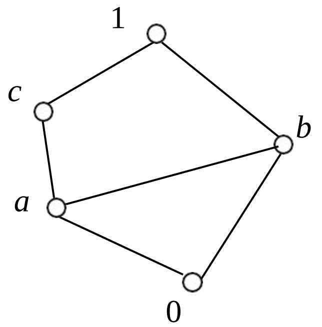
Complement of c does not exist.

$\therefore$ The given distributive lattice is not complemented lattice.

4. Give an example of a lattice that is not complemented.

Solution:

Consider the following lattice.



Since $b \wedge c = a$ and $b \vee c = 1$

$b \wedge a = a$ and $b \vee a = b$

Therefore b does not have any complement.

$\therefore$ The given lattice is not complemented lattice.

Demorgan's laws:

State and prove Demorgan‟s law of lattice.

(or)

Prove that Demorgan‟s laws hold good for a complemented distributive lattice (L, $\wedge$, $\vee$) , viz $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$.

(or)

Let (L, $*$, $\oplus$) be the complemented lattice, then $(a * b)' = a' \oplus b'$ and $(a \oplus b)' = a' * b'$.

Proof:

We note that $a \vee a' = 1,\ a \wedge a' = 0,\ 1 \vee x = 1\ \&\ 0 \wedge x = 0$.

First to prove: $(a \vee b)' = a' \wedge b'$.

It is enough to prove that $(a \vee b) \wedge (a' \wedge b') = 0$ and $(a \vee b) \vee (a' \wedge b') = 1$.

Consider $(a \vee b) \wedge (a' \wedge b')$ $= (a' \wedge b') \wedge (a \vee b)$ (Commutative law)

$= (a' \wedge b' \wedge a) \vee (a' \wedge b' \wedge b)$ (Distributive law)

$= ((a' \wedge a) \wedge b') \vee (a' \wedge (b' \wedge b))$

(Associative and Commutative law)

$= (0 \wedge b') \vee (a' \wedge 0)$ (Since $a \wedge a' = 0$)

$= 0 \vee 0 = 0$ (Since $a \wedge 0 = 0$)

Now, $(a \vee b) \vee (a' \wedge b') = (a \vee b \vee a') \wedge (a \vee b \vee b')$ (Distributive law)

$= (a \vee a' \vee b) \wedge (a \vee b \vee b')$

(Associative and Commutative law)

$= (1 \vee b) \wedge (a \vee 1)$ (Since $a' \vee a = 1$)

$= 1 \wedge 1 = 1$ (Since $1 \vee a = 1$)

$\therefore\ (a \vee b)' = a' \wedge b'$.

Next to prove: $(a \wedge b)' = a' \vee b'$.

It is enough to prove that $(a \wedge b) \vee (a' \vee b') = 1$and $(a \wedge b) \wedge (a' \vee b') = 0$.

Consider $(a \wedge b) + (a' \vee b')$ $= (a' \vee b') \vee (a \wedge b)$ (Commutative law)

$= (a' \vee b' \vee a) \wedge (a' \vee b' \vee b)$ (Distributive law)

$= (a' \vee a \vee b') \wedge (a' \vee b \vee b')$ (Since $a' \vee a = 1$)

$= (1 \vee b') \wedge (a' \vee 1)$ (Since $1 \vee a = 1$)

$= 1 \wedge 1 = 1$

Now, $(a\wedge b)\wedge(a'\vee b') = (a\wedge b\wedge a')\vee(a\wedge b\wedge b')$ (Distributive law)

$= ((a\wedge a')\wedge b)\vee(a\wedge(b\wedge b'))$ (Associative and Commutative law)

$= (0\wedge b)\vee(a\wedge 0)$ (Since $a\wedge a' = 0$)

$= 0\vee 0$ (Since $a\wedge 0 = 0$)

$= 0$

$\Rightarrow (a\wedge b)' = a'\vee b'$.

Example: In a complemented, distributive lattice, show that the following are equivalent. $\leq \Leftrightarrow \wedge' = 0 \Leftrightarrow {}'\vee b = 1 \Leftrightarrow {}'\leq'$

Solution:

(i) $\Rightarrow$ (ii)

Let $\leq$

Then $\wedge =$ and $\vee = b$ $\rightarrow (1)$

$\wedge' = ((\wedge b)\wedge {}')$

$= (\wedge b \wedge {}')$

$= \wedge 0$

$= 0$

Hence $\leq \Leftrightarrow \wedge' = \mathbf{0}$

(ii) $\Rightarrow$ (iii)

Let $\wedge' = 0$

Take complement on both sides, we've $(\wedge')$

$' = 0'$

$'\vee b = 1$

Hence $\wedge\ \mathbf{' = 0} \Rightarrow \mathbf{'\vee\ b = 1}$

(iii) $\Rightarrow$ (iv)

Let $'\vee b = 1$

$\Rightarrow ('\vee b)\wedge' = 1\wedge'$

$\Rightarrow ('\wedge')\vee(b\vee {}') = {}'$

$\Rightarrow ('\wedge')\vee 0 = {}'$ $[\because \wedge' = 0]$

$\Rightarrow {}'\wedge' = {}'$

$\therefore {}' \leq {}'$

Hence $\vee$ **b = 1** $\Leftrightarrow$ **′≤ ′**

(iv) $\Rightarrow$ (i)

Let $' \leq {}'$

Then $' \wedge {}' = {}'$

Take complement on both sides

$\Rightarrow ('\wedge ')' = (')'$

$\Rightarrow \vee b = b$

$\Rightarrow b \geq$

$\Rightarrow a \leq$

Hence **′ ≤ ′** $\Leftrightarrow$ **a** $\leq$

Hence the proof.

Stone's theorem:

Let B be a finite Boolean algebra and let A be the set of all atoms of B . Then prove that the Boolean algebra B is isomorphic to the Boolean algebra $P(A)$, where $P(A)$ is the power set of A.

Proof:

Let B be any finite Boolean algebra.

We will use induction on $|B|$.

We assume that the result is true when $|B| < n$.

To prove: The result is true for $|B| = n$.

Let B be the Boolean algebra with $|B| = n$.

Let "a" be the element of B such that $0 < a < 1$.

Let B_1 be the Boolean algebra $\{0, a\}$ and B_2 be the Boolean algebra $\{0, a'\}$.

We know that $B \;\square\; B_1 \times B_2$.

$\therefore\; 1 \notin B_1 \Rightarrow |B_1| < |B| = n$.

Similarly, $1 \notin B_2 \Rightarrow |B_2| < |B| = n$.

$\therefore$ By induction assumption, there exist a finite set X and Y such that

$B_1 \;\square\; (\rho(X), \cup, \cap)$ and $B_2 \;\square\; (\rho(Y), \cup, \cap)$.

We know that Boolean algebra, $(\rho(X \cup Y), \cup, \cap) \;\square\; (\rho(X), \cup, \cap) \times (\rho(Y), \cup, \cap)$.

Let $Z = X \cup Y$. Then $(\rho(Z), \cup, \cap) \;\square\; (\rho(X), \cup, \cap) \times (\rho(Y), \cup, \cap)$.

$$\square\; B_1 \times B_2$$

$$\square\; B.$$

Therefore $B \;\square\; (\rho(Z), \cup, \cap)$ for a suitable finite set Z.

Now, the smallest number of elements in Boolean algebra is 2 and only Boolean algebra with 2 elements contain only 0 and 1.

If Z = singleton set, then $(\rho(Z), \cup, \cap) \;\square\;$ the Boolean algebra with two elements.

$\therefore$ By induction hypothesis, the result is true for any Boolean algebra.

De Morgan's law:

In a Boolean algebra, prove that $(a+b)' = a' \cdot b'$ and $(a \cdot b)' = a' + b'$

Problems:

1. Show that in a Boolean algebra $ab' + a'b = 0$ if and only if $a = b$.

Solution:

Let $(B, \cdot, +, 0, 1)$ be any Boolean algebra.

Let $a, b \in B$ and $a = b$ .... (1)

To prove: $ab' + a'b = 0$

Now, $ab' + a'b = a \cdot b' + a' \cdot b$

$= a \cdot a' + a' \cdot a$ (Using (1))

$= 0 + 0$ (Since $a' \cdot a = 0$)

$\therefore \quad ab' + a'b = 0.$

Conversely, assume that $ab' + a'b = 0$.

To prove $a = b$.

Now, $a + ab' + a'b = a$ (Left cancelation law)

$\Rightarrow a + a'b = a$ (Absorption law)

$\Rightarrow (a + a') \cdot (a+b) = a$ (Distributive law)

$\Rightarrow \quad 1 \cdot (a+b) = a$ $(a + a' = 1)$

$\Rightarrow a+b = a$ $(1 \cdot a = a)$..(2)

Now, $ab' + a'b + b = b$ (Right cancellation law)

$\Rightarrow ab' + b = b$ (Absorption law)

$\Rightarrow (a + b) \cdot (b'+b) = b$ (Distributive law)

$\Rightarrow \quad (a+b) \cdot 1 = b$ $(b + b' = 1)$

$\Rightarrow a+b = b$ $(1 \cdot a = a)$..(3)

From (2) and (3), we have

$a = a+b = b$

Hence $a = b$.

2. Prove that D_{110}, the set of all positive divisors of a positive integer 110, is a Boolean algebra and find all its sub algebras.

Solution:

D_{110} = Set of all positive divisors of 110.

$\therefore D_{110} = \{1, 2, 5, 10, 11, 22, 55, 110\}$.

Since D satisfies reflexive, antisymmetric and transitive property.

$\therefore$ D is a partial order relation on D_{110}.

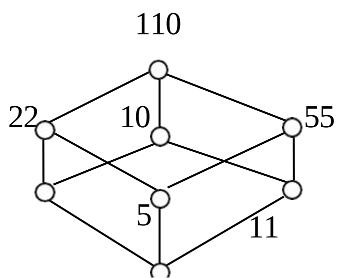
$\therefore (D_{110}, D)$ is a poset.

Here, $a \wedge b = \text{GCD}(a, b)$

$a \vee b = \text{LCM}(a, b), \forall a, b \in D_{110}$, exist.

Therefore $(D_{110}, \wedge, \vee)$ is a lattice.

It's Hasse diagram is 110



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1

Here, least element is 1 and greatest element is 110.

Here, each and every element has complement.

Therefore, $(D_{110}, \wedge, \vee)$ is complemented lattice.

From the Hasse diagram, it is obvious that, it is distributive lattice.

$\therefore (D_{110}, \wedge, \vee)$ is a Boolean algebra.

The sub Boolean algebras are

(i). $\{0, 1\} \equiv \{1, 110\}$,

(ii). $\{1, 2, 5, 10, 11, 22, 55, 110\}$ and

(iii). $\{a, a', 1, 110\}, \forall a \in D_{110}$.

3. Simplify the Boolean expression $a' \cdot b' \cdot c + a \cdot b' \cdot c + a' \cdot b' \cdot c'$.

4. Show that in any Boolean algebra $(a + b)(a' + c) = ac + a'b + bc$.

5. What values of the Boolean variables x and y satisfy $xy = x + y$?

6. In any Boolean algebra, prove that the following statements are equivalent:

   (1) a + b = b, (2) a . b = a, (3) a′ + b = 1 and (4) a . b′ = 0.

7. In a Boolean algebra, show that the following statements are equivalent for any a and b.

   (i) $a+b=b$ (ii) $a \cdot b=a$ (iii) $a'+b=1$ (iv) $a \cdot b'=0$ (v) $a \leq b$

Solution:

First to prove (i) $\Rightarrow$ (ii)

Assume that $a+b=b$

By absorption law, $a = a \cdot (a+b)$

$= a \cdot b$

$\therefore a \cdot b=a$

Next to prove (ii) $\Rightarrow$ (iii)

Assume that $a \cdot b=a$

$a'+b = (a \cdot b)'+b$

$= a'+b'+b$ (De Morgan‟s law)

$a'+b = a'+1$ (Since $a'+a = 1$)

$\therefore a'+b = 1$ (Since $a'+ 1= 1$)

Next to prove (iii)$\Rightarrow$(iv)

Assume that $a^{\text{cc}}+b=1$

$(a^{\text{cc}} + b)' = (1)' = 0$

$a \cdot b' = 0$ (De Morgan‟s law)

Next to prove $(iv) \Rightarrow (v)$

Assume that

$a \cdot b' = 0$ Then

$a \cdot b = a \cdot b + 0$

$= a \cdot b + a \cdot b'$

$= a \cdot (b + b')$ (Distributive law)

$a \cdot b = a$

$\Rightarrow a \leq b$

Finally to prove $(v) \Rightarrow (i)$

Assume that $a \leq b$

$a + b = b$

and $a \cdot b =$

$a\, a + b=$

$a \cdot b + b$

$= a \cdot b + 1 \cdot b$

$= (a + 1) \cdot b$ (Distributive law)

$=1 \cdot b$ (Since $a+ 1= 1$)

$=b$ (Since $1 \cdot a = a$)

Hence $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i)$.

UNIT V

LINEAR PROGRAMMING

5.0 Introduction

Undoubtedly, linear optimization models are among the most commercially successful applications of operation Research. In any organization, supply of resources being limited, its management must find the best allocation of its resources in order to maximize the profit, or minimise the loss and utilize the production capacity to the maximum extent. Linear programming is used as a mathematical modeling technique to optimise the usage of limited resources.

Linear programming as an economic tool was first developed and applied in 1947 by Prof. Dantzig. After the II World War, the application of linear programming has been extended and developed particularly in the economic field. Linear programming is mainly concerned with the problem of allocating limited resources among competitive activities in an optimal manner.

A successful application of LP has been extended in the areas of Military, Agriculture, Transportation, Economics, Health Systems and Behavioral and Social Science. Now a days, the usefulness of the technique is enhanced by the availability of highly efficient Computer codes.

5.0.1 DEFINITIONS OF LINEAR PROGRAMMING

The word ' linear' is used to describe the relationship among two or more variables, which are directly proportioned. For example, if doubling or tripling the production of a product will exactly double or triple the profit and required resource, then it is a linear relationship. The word programming' means planning of activities in a manner that achieves optimum results.

According to Prof. Dantzig, "the interrelations between activities of a large organization be viewed as a linear programming type model and optimising programme determined by minimising a linear objective function"

Technically, linear programme may be formally defined as "a method of optimising (that is maximising or minimising) a linear function with a number of constraints (limitations) expressed in the form of inequalities"

Mathematically, the problem of linear programming may be stated as "one of optimising a linear objective function, under the given constraints and non-negativity of some variable also”.

"Linear programming methods are a technique for choosing the best alternative from a set of feasible alternatives, in situations where the objective functions as well as the constraints are expressed as linear mathematical functions”.

Linear programming involves creating mathematical models to represent problems where the goal is to optimize (maximize or minimize) a linear objective function, subject to a set of linear constraints.

Objective Function: A linear equation that needs to be maximized or minimized.
Example: Maximize $Z = 3x + 4y$.

Constraints: A set of linear inequalities or equations that limit the values of the decision variables. Example: $2x + y \leq 6$.

Decision Variables: The variables that affect the outcome of the objective function, typically representing quantities of products, resources, etc. Example: x and y in the objective function.

Solution Techniques in Linear Programming

There are several methods to solve linear programming problems. The common ones are the Graphical Method, Simplex Method, Big M Method, and Two Phase Method.

Summary of Methods:

1. **Graphical Method**: Best for problems with two variables, visually identifies the optimal solution by examining the feasible region.
2. **Simplex Method**: The most widely used algorithm for large-scale linear programming problems, works for multiple variables.
3. **Big M Method**: Useful when dealing with infeasible initial conditions or inequality constraints.
4. **Two Phase Method**: An alternative to the Big M Method, more numerically stable in practice.

These methods offer a range of tools for solving linear programming problems based on the problem's structure and requirements.

5.0.2 APPLICATIONS OF LINEAR PROGRAMMING

All organisations have to make decisions about how to allocate scarce resources to achieve the orgaisation's goals. Each organisation is attempting to achieve some objective (maximise rate of return, maximise exposure at least cost, maximise profits, minimise cost) with constrained resources (deposits, client advertising budget, available machine time, ingredients) by applying linear programming.

1. **Banking Industry** A bank wants to allocate its funds to achieve the highest possible return. It must operate within the liquidity limits set by regulatory agencies, and it must maintain Sufficient flexibility to meet the loan demands of its customers.

2. **Advertising Industry** An advertising agency wants to achieve the best possible exposure for its client's product at the lowest possible advertising cost. There are a dozen magazines in which it can advertise each one with different advertising rates and differing readership.

3. **Manufacturing Industry** A furniture manufacturer wants to maximise its profits. It has definite limits on the production time available in its three departments as well as delivery commitments to consumers.

4. **Food Industry** A food economist in a developing country wants to prepare a high-protein food mixture at the lowest possible cost. Suppose there are 15 ingredients from which protein can be extracted, and each of these is available in different quantities at different prices. Linear programming is highly helpful in determining the appropriate mixture of these ingredients**.**

5. **Production Management** L P can be applied for determining the optimal product mix to make best use of the machine and man-hours available. It is also used for product smoothing and assembly line balancing.

6. **Personnel Management** Linear programming enables the personnel manager to analyse problems related to recruitment, selection, training, and development of manpower, and to determine the minimum number of employees required in various shifts to meet time varying workloads.

7. **Marketing Management** Linear programming helps a Travelling salesman in finding the Shortest route for his tour. It is also used to determine the optimal distribution schedule for transporting the product from warehouses to various market locations in such a way that the total transportation cost is minimal.

8. **Financial Management** Linear programming helps in the selection of specific investment alternatives from among a wide variety of alternatives.

9. **Agriculture** Linear Programming can be applied in Agricultural planning for the allocation of limited resources, such as labour , water supply, and working capital etc., in a manner that maximises net revenue.

10. **Military Purposes** Usually LP is used in military organisations for selecting an air weapons system against the enemy and at the same time minimising the amount of aviation gasoline used.
11. **Other Applications** Other applications of Linear Programming include the areas of Administration, education, inventory control, fleet utilisations, hospitals, awarding contracts, capital budgeting etc.

5.0.3 ASSUMPTIONS OF LINEAR PROGRAMMING

Practically speaking, objectives and constraints are linear. Hence, linear programming problems are based on the following assumptions:

1. **Proportionality** Proportionally is an assumption about both the objective function and functional constraints. The contribution of each activity to the value of the objective function Z is proportional to the level of the activity Xj, as represented by the CjXj term in the objective function. Similarly, the contribution of each activity to the left-hand side of each functional constraint is proportional to the level of the activity Xj, as represented by the a_{ij} x_j term in the constraint. This means that the constraints increase or decrease Proportionately to the level of each activity. This condition represents constant returns to scales rather than economies or diseconomies of scale.

2. **Additivity** Every function in a linear programming model, whether the objective function or the function on the left-hand side of a functional constraint, is the sum of the individual contributions of the respective activities. It implies that the total contribution of all activities to the constraints is equal to the sum of the contributions for each activity individually. In other words, the whole is equal to the sum of its part.

3. **Divisibility** This assumption concerns itself with the values allowed for the decision variables. Decision variables in a linear programming model are allowed to have any values, including non-integer values, that satisfy the functional and non-negativity constraints**.**

4. **Certainty** Certainty is concerned with the parameters of the model, namely, the coefficient in the objective function C_j, the coefficients in the functional constraints a_{ij}, and the right-hand sides of the functional constraints b_j. The value assigned to each parameter of a Linear Programming model is assumed to be a known constant. This means that each coefficient (C_j,a_{ij},b_j) is fixed and known with certainty.

5.1 Graphical Method

5.1.1 Graphical Method

The graphical method is a simple and intuitive way to solve linear programming problems with two decision variables (i.e., problems in two dimensions). It involves the following steps:

1. **Plot the constraints**: Represent each constraint as a straight line on a graph.
2. **Identify the feasible region**: The feasible region is the area where all constraints are satisfied. It is typically a polygon.
3. **Objective Function**: Plot the objective function as a straight line and move it parallel to itself towards the optimal point.
4. **Find the optimal solution**: The optimal solution is at one of the vertices (corner points) of the feasible region. Calculate the objective function at each vertex to determine the maximum or minimum value.

Example:

Maximize Z=3x+4y

Subject to:

$2x + y \leq 6$

$x + 2y \leq 6$,

$x \geq 0$, $y \geq 0$.

5.1.2 Procedure for solving LPP By Graphical a Method:

Step 1 :

Consider each inequality constraint as an equation.

Step 2 :

Plot each equation on the graph, as each equation will geometrically represent a straight line.

Step 3 :

Mark the region. If the inequality constraint corresponding to that line is ≤, then the region below the line lying in the first quadrant is shaded. For the inequality constraint ≥ sing, the region above the line in the first quadrant is shaded. The points lying in the common region will satisfy all the constraints simultaneously. The common region thus obtained is called 'feasible region '

Step 4 :

Assign a arbitrary value, say zero , to the objective function.

Step 5 :

Draw a straight line to represent the objective function with the arbitrary value.

Step 6 :

Stretch the objective function line till the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin , passing through at least one corner of the feasible region, In the optimization case, this line will stop nearest to the origin, passing through at least one corner of the feasible region.

Step 7 :

Find the coordinates of the extreme points selected in step 6 and find the maximum or minimum value of Z.

Note: As the optimal values occur at the corner points of the feasible regain, it is enough to calculate the value of the objective function of the corner points of the feasible region and select the one that gives the optimal solution. That is in the case of maximization problem, the optimal point corresponds to the corner point at which the objective function has a maximum value, and in the case of minimization, the optimal solution is the corner point which gives the minimum value for the objective function.

Example 1:

Solve the following LPP by graphical Method.

Minimize $z = 20x_1 + 10x_2$

Subject to,

$$x_1 + 2x_2 \geq 40$$
$$3x_1 + x_2 \geq 30$$
$$4x_1 + 3x_2 \geq 60$$
$$x_1, x_2 \geq 0$$

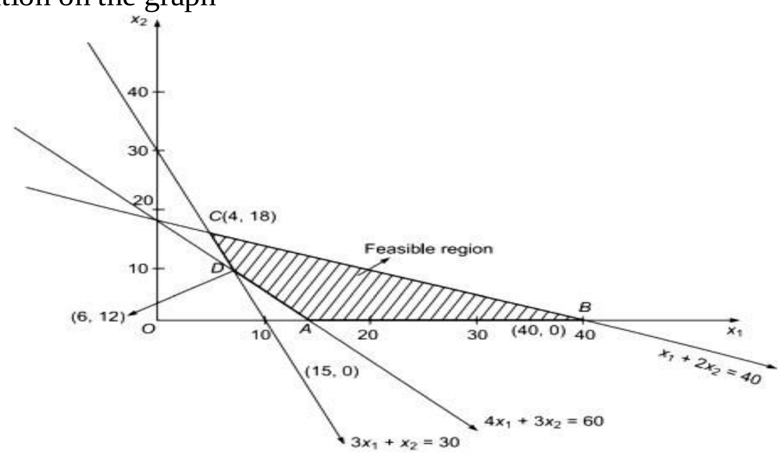
Solution:

Replace all the inequalities of the constraints by equation

$$x_1 + 2x_2 = 40 \quad If \quad x_1 = 0 \Rightarrow x_2 = 20$$
$$If \quad x_2 = 0 \Rightarrow x_1 = 40$$

$$x_1 + 2x_2 = 40 \quad passes \ through \ (0, 20)(40, 0)$$
$$3x_1 + x_2 = 30 \quad passes \ through \ (0, 30)(10, 0)$$
$$4x_1 + 3x_2 = 60 \quad passes \ through \ (0, 20)(15, 0)$$

Plot each equation on the graph



The feasible region is ABCD. C and D are points of intersection of lines.

C intersect $x_1 + 2x_2 = 40, 3x_1 + x_2 = 30$

D intersect $4x_1 + 3x_2 = 60, 3x_1 + x_2 = 60$

corner points	*value of* $z = 20x_1 + 10x_2$
$A(15, 0)$	300
$B(40, 0)$	800
$C(4, 18)$	260
$D(6, 12)$	240

The Minimum Value of Z occurs at $D(6,12)$ .Hence the optimal solution is $x_1 = 6, x_2 = 12$

Example 2 :

Find the maximum value of $z = 5x_1 + 7x_2$

Subject to, the constraints,

$$x_1 + x_2 \leq 4$$
$$3x_1 + 8x_2 \leq 24$$
$$10x_1 + 7x_2 \leq 35$$
$$x_1, x_2 \geq 0$$

Solution :

Replace all the inequalities of the constraints by forming equations

$x_1 + x_2 = 4$ *passes through* $(0, 4)(4, 0)$

$3x_1 + 8x_2 = 24$ *passes through* $(0, 3)(8, 0)$

$10x_1 + 7x_2 = 35$ *passes through* $(0, 5)(3.5, 0)$

Plot each equation on the graph

Graph1

The feasible region is OABCD. B and C are the points of intersection of lines

B intersect $x_1 + x_2 = 4, 10x_1 + 7x_2 = 35$

C intersect $3x_1 + 8x_2 = 24, x_1 + x_2 = 4$

On solving we get,

B = (1.6 , 2.3)

C = (1.6, 2.4)

$corner\ po\text{int}\ s$	$value\ \ of\ z = 5x_1 + 7x_2$
$0(0, 0)$	0
$A(3.5, 0)$	17.5
$B(1.6, 2.3)$	24.1
$C(1.6, 2.4)$	24.8
$D(0, 3)$	21

The Maximum Value of Z occurs at $C(1.6, 2.4)$ and the optimal solution is $x_1 = 1.6, x_2 = 2.4$.

Example 3 :

A company produces 2 types of hats A and B .Every hat A requires twice as much labour time as the second hat B. If the company produces only hat B then it can produce a total of 500 hats per day . The market limits daily sales of hat A and B to 150 and 250 respectively. The profits on hat A and B are Rs. 8 and Rs.5 respectively. Solve graphically to the optimal solution.

Solution:

Let $X_1$and X_2 be the number of units of type A and type B hats respectively.

Max $z = 8x_1 + 5x_2$

Subject to ,

$$2x_1 + x_2 \le 500$$
$$x_1 \le 150$$
$$x_2 \le 250$$
$$x_1, x_2 \ge 0$$

Replace all the inequalities

$2x_1 + x_2 = 500$ $passes\ \ through$ $(0, 500)(250, 0)$

$x_1 = 150$ $passes\ through$ $(150, 0)$

$x_2 = 250$ $passes\ through$ $(0, 250)$

We make the region below the lines lying in the first quadrant as the inequality of the constraints are ≤. The feasible region is OABCD. B and C are the points of intersection of lines.

$$2x_1 + x_2 = 500$$

$$x_1 = 150 \,(B \text{ intersect}) \text{ and}$$

$$x_2 = 250 \,(C \text{ intersect})$$

$$B = (150, 200)$$

$$C = (125, 250)$$

Plot the equation on the graph,

Graph2

corner points	value of $z = 8x_1 + 5x_2$
$O(0,0)$	0
$A(150,0)$	1200
$B(150,200)$	2200
$C(125,250)$	2250
$D(0,250)$	1250

The Maximum Value of Z attained at $C(125,250)$

The optimal solution is $x_1 = 125, x_2 = 250$. The Company should produce 125 hats of type A and B 250 hats of type B in order to get the maximum profits of Rs. 2250.

Example 4 :

By graphical method solve the following LPP

Find the maximum value of $z = 3x_1 + 4x_2$

Subject to,

$$5x_1 + 4x_2 \;\square\; 200$$
$$3x_1 + 5x_2 \;\square\; 150$$
$$5x_1 + 4x_2 \;\square\; 100$$
$$8x_1 + 4x_2 \;\square\; 80$$
$$x_1, x_2 \;\square\; 0$$

Solution:

Replacing the inequality by equality

$5x_1 + 4x_2 = 200$ *passes through* $(0, 50), (40, 0)$

$3x_1 + 5x_2 = 150$ *passes through* $(0, 30), (50, 0)$

$8x_1 + 4x_2 = 80$ *passes through* $(0, 20), (10, 0)$

$5x_1 + 4x_2 = 100$ *passes through* $(0, 25), (20, 0)$

graph3

corner points	*value of* $z = 3x_1 + 4x_2$
$O(20, 0)$	60
$A(40, 0)$	120
$B(30.8, 11.5)$	138.4
$C(0, 30)$	120
$D(0, 25)$	100

The Maximum Value of Z attained at $B(30.8, 11.5)$

The optimal solution is $x_1 = 30.8, x_2 = 11.5$

Example 5:
Use graphical method solve the LPP
Find the maximum value of $z = 6x_1 + 4x_2$
Subject to,

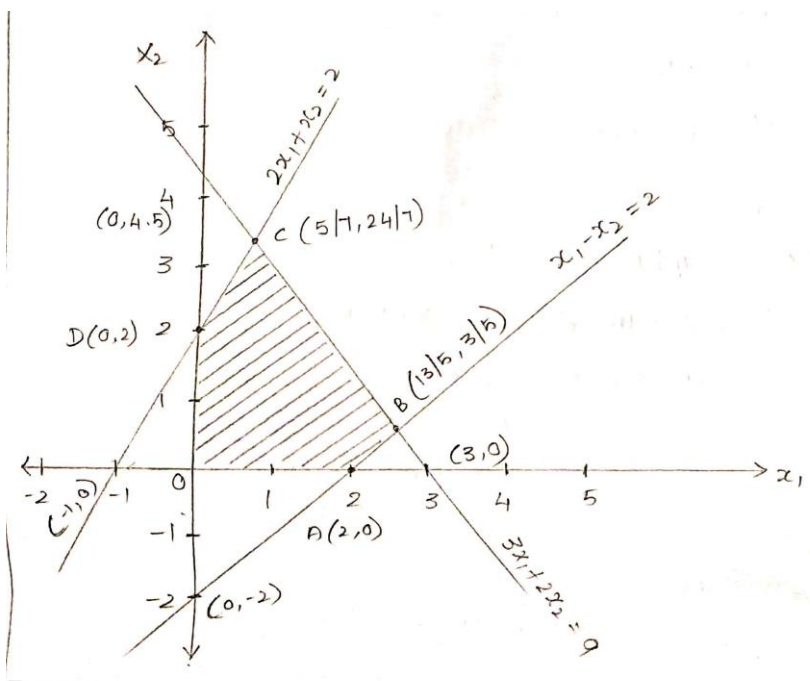
$$-2x_1 + x_2 < 2$$
$$x_1 - x_2 \le 2$$
$$3x_1 + 2x_2 \le 9$$
$$x_1, x_2 \ge 0$$

Solution:

Replacing the inequality by equality

$$-2x_1 + x_2 = 2 \quad \textit{passes through} \quad (0,2)(-1,0)$$
$$x_1 - x_2 = 2 \quad \textit{passes through} \quad (0,-2)(2,0)$$
$$3x_1 + 2x_2 = 9 \quad \textit{passes through} \quad (0,4.5)(3,0)$$

Plot the graph



corner points	*value of* $z = 6x_1 + 4x_2$
$O(0,0)$	0
$A(2,0)$	12
$B\left(\frac{13}{5}, \frac{3}{5}\right)$	$\frac{78+12}{5} = \frac{90}{5} = 18$
$C\left(\frac{5}{7}, \frac{24}{7}\right)$	$= \frac{126}{7} = 18$
$D(0,2)$	$= 8$

The Maximum Value of Z is attained at $B\left(\frac{13}{5}, \frac{3}{5}\right)$ *or at* $\left(\frac{5}{7}, \frac{24}{7}\right)$

The optimal solution is $x_1 = \frac{13}{5}, x_2 = \frac{3}{5}$ *or* $x_1 = \frac{5}{7}, \; x_2 = \frac{24}{7}$

Example 6:

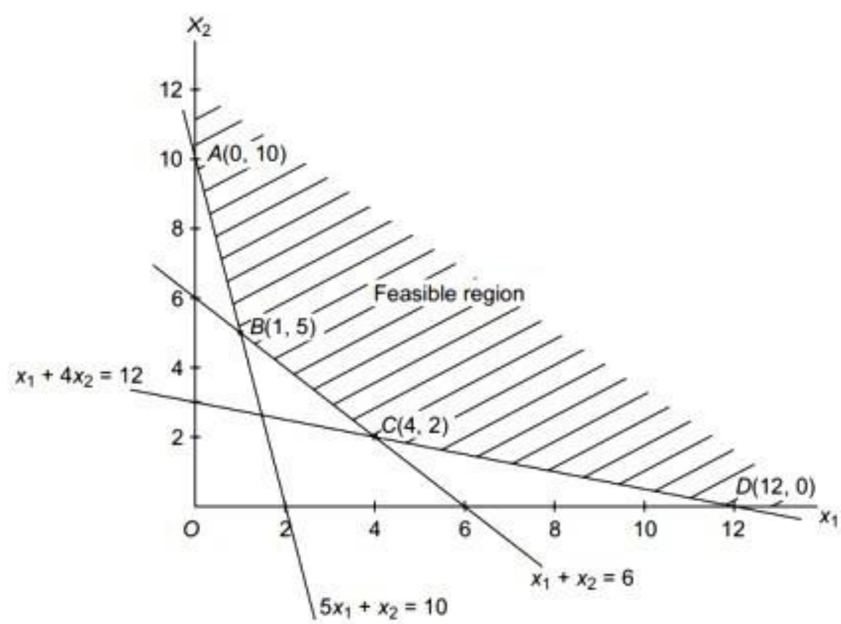
Use graphical method to solve the LPP

Find the maximum value of $z = 3x_1 + 2x_2$

Subject to,

$$5x_1 + x_2 \geq 10$$
$$x_1 + x_2 \geq 6$$
$$x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

Solution:



corner points	value of $z = 3x_1 + 2x_2$
$A(10,0)$	20
$B(1,5)$	13
$C(4,2)$	16
$D(12,0)$	36

Since the minimum Value of Z is attained at $B(1,5)$

The optimal solution is $x_1 = 1, x_2 = 5$

Some more cases:

There are some linear programming problems which may have.

[i] a unique optimal solution

[ii] an infinite number of optimal solution

[iii] an unbounded solution

[iv] no solution

Example 7:

Use graphical method solve the LPP

Find the maximum value of $z = 100x_1 + 40x_2$

Subject to,

$$5x_1 + 2x_2 \leq 1000$$

$$3x_1 + 2x_2 \leq 900$$

$$x_1 + 2x_2 \leq 500$$

$$x_1, x_2 \geq 0$$

Graph4

corner points	*value of* $z = 100x_1 + 40x_2$
$O(0,0)$	0
$A(200,0)$	20000
$B(125,187.5)$	20000
$C(0,250)$	10000

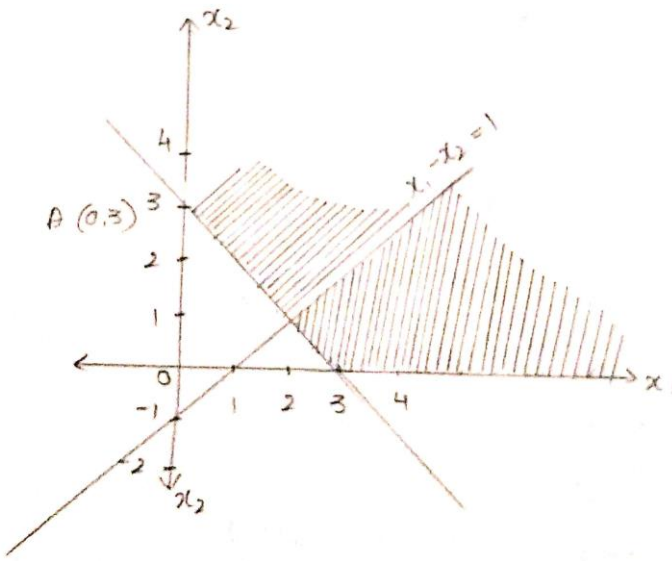
The Maximum Value of Z occurs at two verities A and B .
Thus, there is infinite number of optimal solutions for the LPP

Example 8:
Solve the following LPP.
Maximum value of

$$z = 3x_1 + 2x_2$$
$$x_1 - x_2 \leq 1$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

Solution:

The solution space is unbounded. In fact the maximum size of z occurs at infinity. Hence, the problem has an unbounded solution.
No feasible solution.



Example 9:

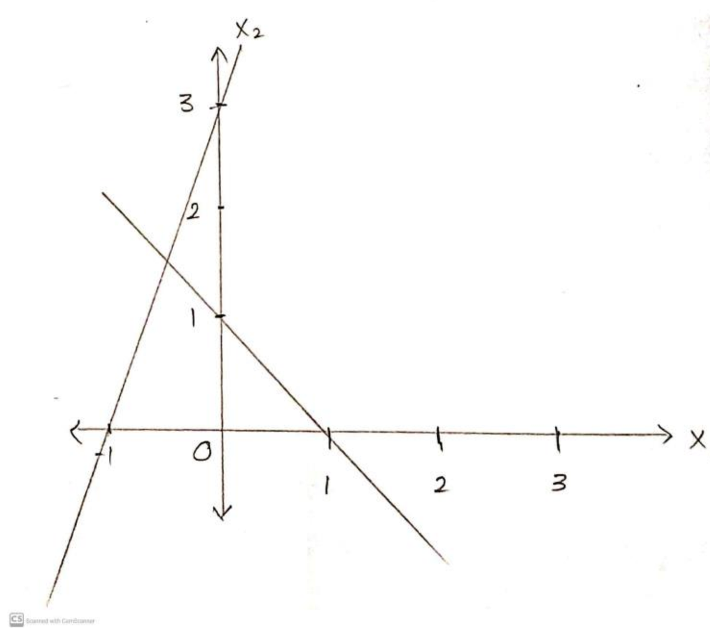
Solve the following LPP.
Maximum value of

$$z = x_1 + x_2$$

$$x_1 + x_2 \leq 1$$
$$-3x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

Solution:

There being no point (x_1, x_2) common to both the shaded regions, we cannot find a feasible region for this problem. So the problem cannot be solved. Hence, the problem has no solution.



Exercises

1. Use graphical method solve the LPP .Find the maximum value of $z = 3x_1 + 2x_2$

   Subject to,

$$-2x_1 + x_2 \leq 1$$
$$x_1 \leq 2$$
$$x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Answer: Z= 8, $x_1 = \frac{2}{3}, x_2 = \frac{7}{3}$

2. Use graphical method solve the LPP .Find the maximum value of $z = 4x_1 + 3x_2$

Subject to,

$$3x_1 + 4x_2 \le 24$$
$$8x_1 + 6x_2 \le 48$$
$$x_1 \le 5,\ x_2 \le 6$$
$$x_1, x_2 \ge 0$$

Answer: Z= 24, $x_1 = 5, x_2 = \frac{4}{3}$

3. Use graphical method solve the LPP .Find the maximum value of $z = 12x_1 + 25x_2$

Subject to,

$$12x_1 + 3x_2 \le 36$$
$$15x_1 - 5x_2 \le 30$$
$$x_1, x_2 \ge 0$$

Answer: Z= 80, $x_1 = \frac{5}{2}, x_2 = 2$

4. Use graphical method solve the LPP .Find the minimum value of $z = 6x_1 + 8x_2$

Subject to,

$$2x_1 + 3x_2 \ge 16$$
$$4x_1 + 2x_2 \ge 16$$
$$x_1, x_2 \ge 0$$

Answer: Z= 24, $x_1 = 4, x_2 = 0$

5. Use graphical method solve the LPP .Find the maximum value of $z = 20x_1 + 10x_2$

Subject to,

$$10x_1 + 5x_2 \le 50$$
$$6x_1 + 10x_2 \le 60$$
$$4x_1 + 12x_2 \le 48$$
$$x_1, x_2 \ge 0$$

Answer: Z= 100, $x_1 = 5, x_2 = 0$

5.2 Simplex Method

5.2.1 Introduction

Simplex method is an iterative produce for solving an LPP in a finite number of steps. It provides an algorithm, which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more, as the case may be, than at the previous vertex. This procedure is repeated and since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of unbounded solution.

The Simplex Method is an iterative algorithm used to find the optimal solution for linear programming problems, especially when there are more than two decision variables.

Steps:

1. **Convert the problem into standard form**: The problem must be in a form where all constraints are equalities, and all variables are non-negative.
2. **Set up the initial Simplex tableau**: This is a matrix representation of the problem where the objective function and constraints are written in terms of basic and non-basic variables.
3. **Iterate**: Perform pivoting steps where you choose an entering variable (non-basic) and a leaving variable (basic). Then, perform row operations to update the tableau.
4. **Repeat until optimal**: Continue pivoting until no further improvement in the objective function can be made (i.e., no positive entries in the objective function row).

The Simplex method is efficient and works for problems with a large number of variables and constraints.

5.2.2 Definitions:

Cost vector:

Let X_B be a basic feasible solution to the LPP

Max $z = CX$

Subject to AX=b and $X \geq 0$

Such that it satisfies $X_B = B^{-1}b$.

Where B is the basis matrix formed by the column of basic variables.

The vector

$C_{Bj} = \left(C_{B1}, C_{B2}, \ldots\ldots, C_{Bm}\right)$, where C_{Bj} are the components of C associate with the basic variables, called the cost vector associated with the basic feasible solution X_B

5.2.3 Evaluation:

Let X_B be a basic feasible solution to the LPP

Max $z = CX$

Subject to AX=b and $X \geq 0$

Let C_B be the cost vector corresponding to X_B . For each column vector a_j is A_1 , which is not a column vector of B, let

$$a_j = \sum_{i=1}^{m} a_{ij} b_i \ .$$

Then the number

$$Z_j = \sum_{i=1}^{m} C_{Bj} a_{ij}$$

is called the Evaluation corresponding to a_j and the number $\left(Z_j - C_j\right)$ is called the net evolution corresponding to j.

5.2.4 Simplex Algorithm:

For the solution of any LPP by simplex algorithm, the existence of an initial basic feasible solution is always assumed. Various steps for the computations of an optimum solution are as follows.

Step: 1

Check whether the objective function of the given LPP is to be maximized [or] minimized . If it is to be minimized then we convert it into a problem of maximization by

$$Min\ z = -\max\left(-z\right)$$

Step: 2

Check whether all $b_i = \left(i = 1, 2, ..., m\right)$ are positive. If any b_i is negative then multiply the in equation of the constraint by -1. So as to get all b_i to be positive.

Step: 3

Express the problem in standard form by introducing slack/ surplus variables to convert the inequality constraints into equations.

Step: 4

Obtain an initial basic feasible solution to the problem in the form $X_B = B^{-1}b$ and put it in the first column of the simplex table. Form the initial simplex table as given below:

Step: 5

Compute the net evaluation $Z_j - C_j$ by using the relation $Z_j - C_j = C_B\left(a_j - c_j\right)$

Examine the sign of $Z_j - C_j$

[i] If all $Z_j - C_j \geq 0$, then the initial basic feasible solution X_B is an optimum basic. Feasible solution.

[ii] If at least one $Z_j - C_j < 0$, then proceed to next step as the solution is not optimal.

Step: 6

[To find the entering variables, [ie] key column]. If there are more than one negative $Z_j - C_j$,choose the most negative of them. Let it be $Z_r - C_r$. For some j = r. This gives the entering variable X_r and is indicated by an arrow at the bottom of the r^{th} column. If there are note than one variables having the same most negative $Z_j - C_j$ then , any one of them can be selected arbitrary as the entering variable.

[i] If all $a_{ir} \leq 0 \left(i = 1, 2, ..., m\right)$ then there an unbounded solution to the given problem.

[ii] If at least one $a_{ir} > 0 \left(i = 1, 2, ..., m\right)$ then the corresponding vector X_r enters the basis.

Step: 7

[To find the leaving variable or key row]

Compute the ration $\left(X_{Bi} / a_{ir}, a_{ir} > 0\right)$.If the minimum of these rations be X_{Bi} / a_{ir}, then choose the variables X_k to leave the basis called the key row and the element at the intersection of key row and key column is called the key element.

Step: 8

Form a new basis by dropping the leaving variable and introducing the entering variable along with the associated value under C_B column convert the leading element to unity by dividing the key equation by the key element and all other elements in its column to zero by using Gauss Elimination Method on the formula.

New element = odd element - [product of elements in key row and column / key element]

Step: 9

Go to step [5] and repeat the procedure until either an optimum solution is obtained or there is an indication of unbounded solution.

Example: 1

A company produces three products A,B,& C. These products required three ores $0_1, 0_2, and\ 0_3$.The maximum quantities of the ores $0_1, 0_2, and\ 0_3$ available are 22 Tonnes ,14 Tonnes and 14 Tonnes respectively . For one Tones of each of these products the are requirements are

Product	O_1	O_2	O_3	Profit per tonne in Thousands
A	3	1	3	1
B	-	2	2	4
C	3	3	0	5

How many tones of each product A,B and C should company produce to maximize the profit ?

Solution:

The mathematical formation of LPP is

Maximize $Z = x_1 + 4x_2 + 5x_3$

Subject to the constraints,

$3x_1 + 3x_3 \leq 22$

$x_1 + 2x_2 + 3x_3 \leq 14$

$3x_1 + 2x_2 \leq 14$ *and* $x_1, x_2, x_3 \geq 0$

Introducing the slack variables $s_1, s_2, s_3 \geq 0$

The standard form of LPP becomes

Maximize $Z = x_1 + 4x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$.

Subject to

$$3x_1 + 3x_2 + s_1 = 22$$
$$x_1 + 2x_2 + 3x_3 + s_2 = 44$$
$$3x_1 + 2x_2 + s_3 = 14$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

An initial basic feasible solution is $x_1, x_2, x_3 = 0 \Rightarrow s_1 = 22, s_2 = 14, s_3 = 14$

I Iteration:

	C_j	1	4	5	0	0	0		
C_B	Basic variable Y_B	X_1	X_2	X_3	S_1	S_2	S_3	Solution value X_B	Minimum Ratio
0	S_1	3	0	3	1	0	0	22	22/3
0	S_2	1	2	3	0	1	0	14	14/3←Mim
0	S_3	3	2	0	0	0	1	14	
	Zj-Cj	-1	-4	-5→Most Negative	0	0	0		

2nd row is the pivot row.

3rd is the pivot element. So $R_2 \to \frac{R_3}{2}$

II Iteration:

	C_j	1	4	5	0	0	0		
C_B	Basic variable Y_B	X_1	X_2	X_3	S_1	S_2	S_3	Solution value X_B	Minimum Ratio
0	S_1	2	-2	0	1	-1	0	8	
0	X_3	1/3	2/3	1	0	1/3	0	14/3	14/2=7
0	S_3	3	2	0	0	0	1	14	14/2=7
	Zj-Cj	2/3	-2/3	0	0	5/3	0		

Example: 2

A firm has available 240, 370 and 18 Kilos of wood, plastic and steel respectively. The firm produces two products A & B .Each unit of product requires 1,3 and 2. Kilos of wood, plastic and steel respectively. The corresponding requirement for each unit of B are 3,4 and 1 respectively . If A Sells for Rs. 4 and B should be produced to obtain the maximum gross income. Formulate the problem mathematically. Solve the problem by simplex method.

Solution:

Product	Wood	Plastic	Steel	Profit
A	1	3	2	Rs.4
B	3	4	1	Rs.6
Availability	240	370	180	

Let X_1 be the units of product A

Let X_2 be the units of product B

The mathematical form of LPP

Maximize $Z = 4x_1 + 6x_2$

Subject to

$$x_1 + 3x_2 \le 240$$

$$3x_1 + 4x_2 \le 370$$

$$2x_1 + x_2 \le 180$$

An introducing the slack variables s_1, s_2 *and* s_3

The mathematical standard form of LPP become select R_3 is pivot row, $R_3 \to \frac{R_3}{3}$

$$R_3 \to \frac{3}{2}\ 1\ 0\ 0\ 0\ \frac{1}{2}\ 7$$

$$R_2 \to R_2 - \frac{R_3}{3}$$

$$R_1 = R_1 + R_3$$

III Iteration :

X_2 enters the basis, s_3 leaves the basis

C_B	Basic variable Y_B	X_1	X_2	X_3	S_1	S_2	S_3	Solution value X_B	Ratio θ
0	S_1	5	0	0	1	-1	1	22	
5	X_3	-2/3	0	1	0	1/3	-1/3	0	
4	X_2	3/2	1	0	0	0	1/2	7	
	Zj-Cj	5/3	0	0	0	5/3	1/3		

All $Z_j - C_j \geq 0$

The optimal solution is obtained

Maximize

$$Z = 28000$$
$$x_2 = 7$$
$$x_1 = 0$$
$$x_3 = 0$$

The Maximum values of Z occurs when, the company should produce 7 tonnes of product . B and none of A [or] C.

$$x_1 + 3x_2 + s_1 = 240$$
$$3x_1 + 4x_2 + s_2 = 370$$
$$2x_1 + x_2 + s_3 = 180$$

I Iteration:

An initial basic feasible solution is $x_1 = x_2 = 0, s_1 = 240, s_2 = 370, s_3 = 180.$

	C_j	4	6	0	0	0		
C_B	Basic variable Y_B	X_1	X_2	S_1	S_2	S_3	Solution value X_B	Ratio θ
0	S_1	1	3	1	0	0	240	240/3=80
0	S_2	3	4	0	1	0	370	376/4=925
0	S_3	2	1	0	0	1	180	180/1=180
	Zj-Cj	-4	-6 Max. negative	0	0	0		

I row is pivot row , 3 is the pivot element X_2 enters the basis. S_1 leaves the basis.

II Iteration:

X_2 enters the basis. S_2 leaves outside the basis.

C_B	Basic variable Y_B	X_1	X_2	S_1	S_2	S_3	Solution value X_B	Ratio
6	X_2	1/3	1	1/3	0	0	80	240
0	S_2	5/3	0	-4/3	1	0	50	30□Mim
0	S_3	5/3	0	-1/3	0	1	100	60
	Zj-Cj	-2 most negative	0	2	0	0		

II Iteration:

X_1 enters the basis. S_1 leaves the basis

	C_j	4	6	0	0	0		
C_B	Basic variable Y_B	X_1	X_2	S_1	S_2	S_3	Solution value X_B	Ratio θ
6	X_2	0	5	1/3	-1	0	350	
4	X_1	1	0	-4/5	3/5	0	30	
0	S_3	0	0	1	-1	1	50	
	Zj-Cj	0	24	-6/5	-18/5	0		

Zj-Cj ≥ 0, The current optional feasible solution is obtained.

The Maximize

$$Z = 1670$$
$$x_1 = 30$$
$$x_2 = 350$$

Example: 3

Find the non – negative values of X_1 , X_2 and X_3

Which maximize $Z = 3x_1 + 2x_2 + 5x_3$

Subject to

$$x_1 + 4x_2 \le 420$$
$$3x_1 + 2x_2 \le 460$$
$$x_1 + 2x_2 + x_3 \le 430$$

Solution:

In the given LPP constraints $\leq$ symbol, by introducing the slack variables s_1, s_2 $and\, s_3$

The standard form of LPP become

Maximize $Z = 3x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$.

$$x_1 + 4x_2 + s_1 = 420$$
$$3x_1 + 0x_2 + 2x_3 + s_2 = 460$$
$$x_1 + 2x_2 + x_3 + s_3 = 430$$

There are 3 equations with 6 variables, the initial basic feasible solutions is obtained by equating 6-3=3 variables to zero.

The initial basic feasible solution is

$s_1 = 420$

$s_2 = 460$

$s_3 = 430$ $\quad (x_1 = x_2 = x_3 = 0 : non - basic)$

I Iteration :

C_B	Y_B	X_1	X_2	X_3	S_1	S_2	S_3	X_B	Ratio
0	S_1	1	4	0	1	0	0	420	
0	S_2	3	0	2	0	1	0	460	460/2=230---Min
0	S_3	1	2	1	0	0	1	430	430/1=430
	Zj-Cj	-3	-2	-5	0	0	0		

II row is pivot row, pivot element is 2. Make pivot element as 1 and the corresponding other element in the same column as zero.

II Iteration

	Cj	3	2	5	0	0	0		
C_B	Y_B	X_1	X_2	X_3	S_1	S_2	S_3	X_B	Ratio
0	S_1	1	4	0	1	0	0	420	420/4=105
5	X_3	3/2	0	1	0	1/2	0	230	
0	S_3	-1/2	2	0	0	-1/2	1	200	200/2=100
	Zj-Cj	9/2	-2	0	0	5/2	0		

Here Zj-Cj =-2, the current basic feasible solution is not optimal.

Here the non- basic feasible variable X_2 enters into the basis and basic variable S_3 leaves the basis

Pivot row R_3. Pivot element is 2. Make it as 1 and the other element in the same column as zero.

II Iteration

C_B	Y_B	X_1	X_2	X_3	S_1	S_2	S_3	X_B	Ratio
0	S_1	2	0	0	1	1	-2	20	
5	X_3	3/2	0	1	0	1/2	0	230	
2	X_2	-1/4	1	0	0	-1/4	1/2	100	
	Zj-Cj	4	0	0	0	2	1	1350	

All $Zj - Cj \geq 0$

The current basic feasible solution is an optimal.

The optimal solution is Maximize $Z = 1350$, $x_1 = 0, x_2 = 100, x_3 = 230$

Example: 4

Apply the simple method to solve the problem

Maximize

$$Z = 100\,x_1 + 200x_2 + 50x_3$$

Subject to

$$5x_1 + 5x_2 + 10x_3 \leq 1000$$
$$10x_1 + 8x_2 + 5x_3 \leq 2000$$
$$10x_1 + 5x_2 \leq 500$$
$$x_1, x_2, x_3 \geq 0$$

Solution:

By introducing the slack variables s_1, s_2 *and* s_3. The standard form of LPP becomes,

Maximize

$$Z = 100\,x_1 + 200x_2 + 50x_3 + 0s_1 + 0s_2 + 0s_3$$

Subject to

$$5x_1 + 5x_2 + 10x_3 + s_1 = 1000$$
$$10x_1 + 8x_2 + 5x_3 + s_2 = 2000$$
$$10x_1 + 5x_2 + s_3 = 500$$

There are 3 equations with 6 variables.

The initial basic feasible solutions is

$$s_1 = 1000$$
$$s_2 = 2000$$
$$s_3 = 500 \qquad (x_1 = x_2 = x_3 = 0, non-basic)$$

I Iteration

C_B	Y_B	X_1	X_2	X_3	S_1	S_2	S_3	X_B	Ratio
0	S_1	5	5	10	1	0	0	1000	1000/5=200
0	S_2	10	8	5	0	1	0	2000	2000/4=250
0	S_3	10	5	0	0	0	1	500	500/5=100
	Zj-Cj	-100	-200	-50	0	0	0		

Most negati

II Iteration :

X_2 enters the basis. S_3 leaves the basis.

C_B	Y_B	X_1	X_2	X_3	S_1	S_2	S_3	X_B	Ratio
0	S_1	-5	0	10	1	0	-1	500	500/10=50
0	S_2	-6	0	5	0	1	-8/5	1200	1200/5=240
200	X_2	2	1	0	0	0	1/5	100	
	Zj-Cj	300	0	-50	0	0	40		

Most negative

III Iteration

C_j			100	200	50	0	0	0		
	C_B	Y_B	X_1	X_2	X_3	S_1	S_2	S_3	X_B	Ratio
	50	X_3	-1/2	0	1	1/10	0	-1/10	50	
	0	S_2	-7	0	0	-1	2	-11/5	1900	
	200	X_2	2	1	0	0	0	1/5	100	

Zj-Cj 275 0 0 5 0 35 22500

Here Zj-Cj $\geq$ 0

The current basic feasible solution is an optimal.
The optimal solution is Maximize

$$Z = 22500$$

$$x_1 = 0, x_2 = 200, x_3 = 50$$

Example: 5

Use simplex method to solve the given LPP

Max $\quad Z = 5x_1 + 3x_2$

Subject to the constraints

$$x_1 + x_2 \leq 2$$

$$5x_1 + 2x_2 \leq 10$$

$$3x_1 + 8x_2 \leq 12, \; x_1, x_2 \geq 0$$

Solution:

Introducing the slack variables s_1, s_2 *and* s_3. The mathematical formation of LPP becomes,

Maximize

$$Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$

Subject to the constraints

$$x_1 + x_2 + s_1 = 2$$

$$5x_1 + 2x_2 + s_2 = 10$$

$$3x_1 + 8x_2 + s_3 = 12$$

Initial basic feasible solution is

$x_1 = x_2 = 0$

$s_1 = 2, s_2 = 10, s_3 = 12$

I Iteration:

C_B	Y_B	X_1	X_2	S_1	S_2	S_3	X_B	Ratio
0	S_1	1	1	1	0	0	2	2
0	S_2	5	2	0	1	0	10	2
0	S_3	3	8	0	0	1	12	4
	Zj-Cj	-5	-3	0	0	0		

Most negative

R_1 is pivot, 1 is pivot element

Make it other element in the zeros.

Column as zero.

II Iteration: X_1 enters the basis, S_1 leaves the basis

C_j 5 3 0 0 0

C_B	Y_B	X_1	X_2	S_1	S_2	S_3	X_B	Ratio
5	X_1	1	1	1	0	0	2	
0	S_2	0	-3	-5	1	0	0	
0	S_3	0	5	-3	0	1	6	
	Zj-Cj	0	2	5	0	0		

All Zj-Cj $\geq$ 0

The solution is an optimal

$$x_1 = 5$$

$$x_2 = 0$$

$$z = 5(2) + 0$$

$$= 10$$

Minimization Type:

Example: 6

Use simplex method to

Minimize $Z = x_1 - 3x_2 + 2x_3$

Subject to

$$3x_1 - x_2 + 2x_3 \leq 7$$

$$-2x_1 + 4x_2 \leq 12$$

$$-4x_1 + 3x_2 + 8x_3 \leq 10 \quad and \quad x_1, x_2, x_3 \geq 0$$

Solution:

The given objectives function is of minimization type, we shall convert it into maximization type as follows.

Maximize [-Z] = Maximize $Z^* = -x_1 + 3x_2 - 2x_3$.

By introducing the slack variables. $s_1, s_2 \ and \ s_3$.

The standard form of LPP becomes,

Maximize $\quad Z^{*} = -x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to constraints

$$3x_1 - x_2 + 2x_3 + s_1 = 7$$
$$-2x_1 + 4x_2 + s_2 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$$

Initial basic feasible solution is given by

$$x_1 = x_2 = x_3 = 0$$
$$s_1 = 7, s_2 = 12, s_3 = 10$$

I Iteration:

C_B	Y_B	X_1	X_3	X_2	S_1	S_2	S_3	X_B	Ratio
0	S_1	3	-1	2	1	0	0	7	
0	S_2	-2	4	0	0	1	0	12	12/4=3
0	S_3	-4	3	8	0	0	1	10	10/3=3.33
	Zj-Cj	1	-3	2	0	0	0		

II Iteration:

C_B	Y_B	X_1	X_3	X_2	S_1	S_2	S_3	X_B	Ratio
0	S_1	5/2	0	2	1	¼	0	10	10/5/2=4
3	X_2	-1/2	1	0	0	¼	0	3	
0	S_3	-5/2	0	8	0	-3/4	1	1	
	Zj-Cj	-1/2	0	2	0	¾	0		

R_1 is pivot, -1/2 is pivot element

Make it as 1 and other element as zeros.

III Iteration:

C_B	Y_B	X_1	X_3	X_2	S_1	S_2	S_3	X_B	Ratio
-1	X_1	1	0	4/5	2/5	1/10	0	4	
3	X_2	0	5	2	1	3/2	0	25	
0	S_3	0	0	10	1	-1/2	1	11	
	Zj-Cj	0	12	16/5	13/5	44/10	0		

All $\quad$ Zj-Cj ≥ 0

The optimal solution is obtained

Maximize

$$Z^* = -4 + 3(25)$$
$$= 71$$
$$x_1 = 4, x_2 = 25, x_3 = 0$$

Maximize Z = - Maximize Z* = -71.

Exercises

1. Use simplex method to

Maximize $Z = 5x_1 + 6x_2 + x_3$

Subject to

$$9x_1 + 3x_2 - 2x_3 \leq 5$$
$$4x_1 + 2x_2 - x_3 \leq 2$$
$$x_1 - 4x_2 + x_3 \leq 3 \; and \; x_1, x_2, x_3 \geq 0$$

Answer: Unbounded

2. Use simplex method to

Maximize $Z = 30x_1 + 40x_2 + 20x_3$

Subject to

$$10x_1 + 120x_2 + 70x_3 \leq 100000$$
$$7x_1 + 10x_2 + 8x_3 \leq 8000$$
$$x_1 + x_2 + x_3 \leq 1000$$
$$x_1, x_2, x_3 \geq 0$$

Answer: Z=32500, $x_1 = 250, x_2 = 625, x_3 = 125$

3. Use simplex method to

Maximize $Z = 10x_1 + 15x_2 + 20x_3$

Subject to

$$2x_1 + 4x_2 + 6x_3 \leq 24$$
$$3x_1 + 9x_2 + 6x_3 \leq 30$$
$$x_1, x_2, x_3 \geq 0$$

Answer: Z=100, $x_1 = 6, x_2 = 0, x_3 = 2$

5.3 Linear Programming

5.3.1 Introduction:

Liner Programming deals with the optimization [maximization or minimization] of a function of variables known as objective function. It is a subject consisting of a set of linear equalities and / or inequalities known as constraints. Linear Programming is a mathematical technique which involves the allocation of limited resources in an optimal manner, on the basis of a given criterion of optimality.

5.3.2 Formulation of LP Problems

The procedure for mathematical formulation of a LPP consists of the following steps:

Step:1

To write down the decision variables of the problem.

Step: 2

To formulate the objective function to be optimized [maximization or minimization] as a linear function of the decision variables.

Step: 3

To formulate the other conditions of the problems such as resource limitation, market. Constraints, interrelation between variables, etc, as linear in equations [or] equations in terms of the decision variables.

Step:4

To add the non – negativity constraints from the considerations so that the negative value of the decision variables do not have any valid physical interpretations.

The objective function, the set of constraints and the non – negative restrictions together from a linear programming problem [LPP]

General Formulation of LPP

The general formulation of the LPP can be stated as follows

In order to find the values of n decision variables $x_1, x_2, \ldots x_n$. to maximize [or] minimize the objective function $z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$

And also satisfy – constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_{n\,\ldots\ldots}b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_{n\,\ldots\ldots}b_2$$

.

.

$$a_{i1}x_1 + a_{i2}x_2 + \ldots\ldots + a_{in}x_{n\,\ldots\ldots}b_i$$

.

.

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_{n\ldots\ldots}b_m$$

Where constraints may be in the form of inequality $\le$ (or) $\ge$ (or) *even* in the form of an equation [=] and finally satisfy the non – negative restriction $x_1 \ge 0, x_2 \ge 0 \ldots\ldots x_n \ge 0$

Matrix Form of LP problem

The LPP can be expressed in the matrix form as follows

Minimize or

Maximize Z = cx --> objective function

Subject to , $Ax(\le = \ge)b$ constraint equation $b > 0, x \ge 0$ Non – negativity restrictions.

$$x = (x_1 x_2 \ldots\ldots x_n)$$

Where, $c = (c_1 c_2 \ldots\ldots c_n)$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{pmatrix}_{m \times 1} \quad A = \begin{bmatrix} a_{11} a_{12} \ldots\ldots + a_{1n} \\ a_{21} a_{22} \ldots\ldots + a_{2n} \\ . \\ . \\ . \\ a_{m1} a_{m2} \ldots\ldots + a_{mn} \end{bmatrix}_{m \times n}$$

Example: 1

Production Allocation Problem

A firm manufactures two types of products A and B and sells them at a profit of Rs. 2 on type A and Rs.3 on type B Each product is proceed on two machines G and H Type A requires one minute of processing time on G and two minute on H ; type B requires one minute on G and one minute on H. The Machine G is available for not more than 6 hours 40 minutes while machine H is available for 10 hours during any working day. Formulate the problem as a linear programming problem.

Formulation

Decision variables Let X_1 be the number of products of type A to be produced and Let X_2 be the number of units of type B to be produced.

Table 2.1

Machine	Time on products Type A [x_1 units]	Type B [x_2 units]	Available time [minutes]
G	1	1	400
H	2	1	600

Objective function : Since the profit on type A is 2 per product $2X_1$ will be the profit on selling X_1 units of type A Similarly $3x_2$ will be the profit on selling X_2 units of type B Therefore, total profit Z on shelling X_1 units of A and X_2 units of B is given by $z = 2x_1 + 3x_2$

Example: 2 Blending Problem

A tirm produces an alloy having the following specifications

[i] Specific gravity ≤ 0.98

[ii] chromium $\geq 8\%$

[iii] melting point $\geq 450^{0}C$

Raw materials' A,B,C having the properties shown in the Table 2.2 can be used to make the alloy

Table 2.2

Property	Properties of raw Materials		
	A	B	C
Specific gravity	0.92	0.97	1.04
Chromium	7%	13%	16%
Melting point	$440^{0}C$	$490^{0}C$	$480^{0}C$

Cost of the various raw materials per unit are: Rs. 90 for A Rs. 280 for B and Rs. 40 for C. Find the property in which A,B,C be used to obtain an alloy of derived properties while the cost of raw materials is minimum.

Hence the objective function is maximize $z = 2x_1 + 3x_2$

Constraints: Since machine G takes 1 minute time on type A and 1 minute time on type B, the total number of minutes required or machine G is given by $x_1 + x_2$ but machine G is available for not more than 6 hours 40 minutes [400 minutes] .Therefore $x_1 + x_2 \leq 400$

Similarly, the total number of minutes required on machine H is given by $2x_1 + 3x_2$ .But machine H is available for 0 hours on a working Day. Therefore $2x_1 + x_2 \leq 600$.

Non – negative restrictions: Since it is not possible to produce negative quantities we have $x_1 \geq 0, x_2 \geq 0$

Hence allocation problem can be represented as

Maximize

$$z = 2x_1 + 3x_2$$
$$x_1 + x_2 \leq 400$$
$$2x_1 + x_2 \leq 600$$
$$x_1 x_2 \geq 0$$

subject to and that total exposure to principal buyers of expensive sports goods is maximized percentages of readers for each magazine are known Exposure in any particular magazine is the number of advertisements placed multiplied by the number of principal buyers. The following data may be used.

Table 2.3

Exposure category	Magazine		
	A	B	C
Readers	1 lakh	0.6 lakh	0.4 lakh
Principal buyers	10%	15%	7%
Cost per advertisement Rs.	5000	4500	4250

The budgeted amount is Rs. 1 lakh for the advertisement. The owner has already decided that magazine A should have to no more than six advertisement and that B and C each have atleast two advertisements formulate a linear programming model for the problem.

Formulation

Decision variables Let X_1 be the number of advertisement placed in magazine A and let X_2 be the number of advertisements placed in Formulation

Decision variables Let the percentage contents of A, B, C be x_1, x_2, x_3, respectively.

Objective function to minimize the cost

$z = 90x_1 + 280x_2 + 40x_3$

Constraints: constraints are imposed by the specification required for the alloy. They are

$$\left.\begin{aligned} 0.92x_1 + 0.97x_2 + 1.04x_3 &\geq 0.98 \\ 7x_1 + 13x_2 + 16x_3 &\geq 8 \\ 440x_1 + 490x_2 + 480x_3 &\geq 450 \\ x_1 + x_2 + x_3 &= 100 \\ x_1 x_2 x_3 &\geq 0 \end{aligned}\right\}$$

Non – negative restrictions

The relations [2.2.1] and [2.2.2] and [2.2.3] together represent the linear programming model for the given problem.

Example: 3 Media Selection problem

The owner of Metro sports wishes to determine how many advertisement to place in three selected monthly magazines' A, B, C. His objective to advertise in such a way that magazine B; X_3 be the number of advertisement placed in magazine C

Objective function:

The objective is to advertise in such a way that hotel exposure to principal buyers of expensive sports good is maximized.

The number of readers who buy magazine A is 1,00,000 and to the readers who have seen the advertisement are 1,00,000 X_1 Similarly the number of advertisement seen by the readers of magazine B is 60,000 X_2 and the number of advertisement seen by the readers of magazine C is 40,000 X_3 : Hence the objective function is

Maximize $x_1, x_2, x_3, x_4 \geq 0$

$$z = 0.10 \times 1,00,000x_1 + 0.15 \times 60,000x_2 + 0.7 \times 40,000x_3$$
$$= 10,000x_1 + 9,000x_2 + 2,800x_3$$

Constraints: Given the budgeting amount is at most Rs.1 Lakh for the advertisement. The lost for all advertisement in the entire three magazines is

$$500x_1 + 4500x_2 + 4250x_3$$

[Bud getting]

$$5000x_1 + 4500x_2 + 4250x_3 \leq 1,00,00$$

Given that the owner have decided that magazine A should not have more than six advertisement and B,C have at least two advertisements each

$x_1 \leq 6x_2 \geq 2x_3 \geq 2$ [Advertising]

Non – negative restrictions

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

The relation from [2.2.4] through [2.2.7] gives the linear programming model for the given problem.

Example 4: **Diet problem:**

A person wants to decide the constituents of a diet which will fulfill his daily requirements of proteins, carbohydrates at the minimum cost. This choice is to be made from four different types of foods. The yield per unit of these foods is given in the Table.

Table 2.4

Food type	**Proteins**	**Yield per unit Fats**	**Carbohydrates**	**Cost per units Rs.**
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum requirement	800	200	700	

Formulate the linear programming model for the problem

Formulation

Decision Variables Let x_1, x_2, x_3, x_4 be the units of food type $1, 2, 3, 4$ respectively

Objective function: Since the cost of these food of type $1, 2, 3, 4$ are Rs. 45,Rs. 40, Rs.85, Rs.65 per unit the total cost is $Rs.45x_1 + 40x_2 + 85x_3 + 65x_4$. As the objective function is to minimize the total cost , the objective function is

Minimize $z = 45x_1 + 40x_2 + 85x_3 + 65x_4$ constraints from the units of food type $1, 2, 3, 4$ he requires

$$3x_1 + 4x_2 + 8x_3 + 6x_4 \qquad \left(protiens / day \right)$$
$$2x_1 + 2x_2 + 7x_3 + 5x_4 \qquad \left(Fats / day \right)$$
$$6x_1 + 4x_2 + 7x_3 + 4x_4 \qquad \left(carbohydrates \ / \ day \right)$$

Since the minimum requirement of these proteins ,fats and carbohydrates are 800,200 and 700 respectively the constraints are

$$3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$$
$$2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$$
$$6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$$

Non – Negativity restrictions

$x_1, x_2, x_3, x_4 \geq 0$

Relations form [2.2.8] through [2.2.10] give the complete formulation of LPP.

Example 5: warehouse problem

A person running a warehouse purchases and sells identical items. The warehouse can accommodate 1,000.such items. Each month, the person can sell any quantity he has in shock. Each month he can buy as much as the likes to have in shock and delivery at the end of month, subject to maximum of 1,000 items. The forecast of purchase and sales prices for the next six months is given below. Table 2.5

Month i	1	2	3	4	5	6
Purchase price Ci [Rs]	12	14	17	19	20	21
Sale price Si [Rs]	13	15	16	20	21	23

If at present he has a shock of 200 items, what should be his policy?

Formulation

Decision Variables Let X_1 and $y_i \left(i = 1, 2, ...6 \right)$ be the number of items purchased and sold for each of the six months

Objective function the objective is to maximize the profit. That is Maximize

$$z = 13y_1 + 15y_2 + 16y_3 + 20y_4 + 21y_5 + 23y_6 - \left(12x_1 + 14x_2 + 17x_3 + 19x_4 + 20x_5 + 21x_6 \right)$$

Constraints The person cannot sell anything that is not is shock .Therefore for each month

$$n = 1, 2, \ldots\ldots 6$$

$$200 + \sum_{j=1}^{n-1} \left(x_i - y_i \right) \geq y_n$$

$$200 + \sum_{j=1}^{n-1} x_j \geq \sum_{j=1}^{n-1} y_j + y_n$$

$$200 + \sum_{j=1}^{n-1} x_j \geq \sum_{j=1}^{n} y_j$$

$$\left.\begin{array}{l} \sum_{j=1}^{n} y_j - \sum_{j=1}^{n-1} x_j \leq 200 \\ \sum_{j=1}^{6} y_j - \sum_{j=1}^{5} x_j \leq 200 \end{array}\right\}$$

Further, since we cannot over shock beyond 1,000 items.

$$200 + \sum_{j=1}^{n} \left(x_j - y_j \right) \leq 1,000$$

$$\sum_{j=1}^{n} x_j - \sum_{j=1}^{n} y_j \leq 800$$

$$\sum_{j=1}^{6} x_j - \sum_{j=1}^{6} y_j \leq 800$$

Non – negativity restrictions

$$x_j \geq 0, y_j \geq 0 \quad for \;\; j = 1, 2, \ldots\ldots 6$$

The equation [2.2.11] – [2.2.14] gives the linear programming model .

Example 6: Product Mix problem

A Firm manufactures three products A,B,C Time to manufacture product A is twice that for B and thrice that for C and to be product in the ration 3:4:5 The revel ant data is given in Table 2:6 if the whole raw material is engaged in manufacturing product A, 1600 units of this product can be product. There is demand for atleast 300,250,200 units of products A,B and C and the profit corned per unit is Rs.50,Rs.40,Rs.70, respectably . Formulate the problem as a linear programming problem.

Table 2.6

Raw material	Requirements per unit product [kg]			Total availabilities [Kg]
	A	B	C	
P	6	5	9	5,000
Q	4	7	8	6,000

Formulation:

To determine the number of units of each of the products A,B,C to be manufactured

Decision Variables: Let x_1, x_2, x_3 be the number of units products A, B, C to be manufactured.

Objective function: To maximize profit

Maximize $z = 50x_1 + 40x_2 + 70x_3$.

For raw material $P, 6x_1 + 5x_2 + 9x_3 \leq 5,000$

For raw material $Q, 4x_1 + 7x_2 + 8x_3 \leq 6,000$

As product B requires $\frac{1}{2}$ and product C requires $\frac{1}{3}$ of the time required for product A. The constraints on number of units manufactured may be expressed as

$$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 \leq 1,600$$

Market

$$\left.\begin{aligned} x_1 &\leq 300 \\ x_2 &\leq 250 \\ x_3 &\leq 200 \end{aligned}\right\}$$

Since products A,B,C are to be product in the ratio 3 : 4 : 5

$$x_1 : x_2 : x_3 = 3 : 4 : 5$$

$$\frac{x_1}{3} = \frac{x_2}{4} \; and \; \frac{x_2}{4} = \frac{x_3}{5}$$

$$4x_1 - 3x_2 = 0 \quad and \quad 5x_2 - 4x_3 = 0$$

Non – negativity restrictions

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

The relation from [2.2.15] through [2.2.19] gives the complete linear programming model .

Example: 7 Product Mix Problem

A Chemical company produces two products X and Y. Each unit of product X requires 3 hours on operation 1 and 4 hours on operation ll. While each unit of product Y requires 4 hours on operation 1 and 5 hours on operation ll. Total available time for operation l and ll are 20 hours and 26 hours respectively. The production of each unit of product Y also result in the production of two units of a by – product Z at no extra cost. Product X sells as profit of Rs. £10 / unit, while Y sells at profit of Rs.20/ unit By – product Z brings a unit profit Rs.6 if sold in case it cannot be sold, the destruction cost is Rs.4 / unit .Fore cents indicate that not move than 5 units of Z can be sold .Determine the qualities of X and Y to be produced keeping Z in mind, so the profit earned in maximum.

Formulation:

To determine the number of units products X,Y,Z to be produced.

Decision Variables: Let the number of units products X,Y,Z produced be x_1, x_2, x_3 respectively .Note that

X_1 = Number of units of Z sold + Number of units of Z destroyed = $X_4 + X_5$ [say]

Objective function: To maximize the profit earned. Objective function [profit function] for products X and Y is linear because their profits are constant irrespective of the number of units produced. A graph between the total profit and quantity produced will be straight line . But a similar graph of Z show that Z is non – linear, since it has slope + 6 for the first part and – 4 for the second However it is piecewise linear, since it is linear in the region [0,5] and [5, y] so splitting X_3 into two parts, namely the number of unit's of Z sold [X_4] and number of units Z destroyed [X_5] the objective function for product Z also linear Thus the objective function is

Maximize $z = 10x_1 + 20x_2 + 6x_3 - 4x_5$.

Constraints : Since each unit of product X requires 3 hours on operation I and 4 hours on operation II each unit of product Y required 4 hours on operation I and 5 hours on operation II the total time available for operation I and II is 20 hours and 26 hours respectively . We have the constraints.

$$\left.\begin{aligned} 3x_1 + 4x_2 &\leq 20 \left(\textit{for operation I} \right) \\ 4x_1 + 5x_2 &\leq 26 \left(\textit{for operation II} \right) \end{aligned}\right\}$$

Since not move then of Z can be sold

$$x_3 \leq 5$$

The production of each unit of product Y also result in two units of by – product Z at no extra cost .so

$$x_2 = 2x_3 = x_4 + x_5$$

Non – negative restriction

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

The relation from [2.2.20] through [2.2.24] gives the complete formulation of linear programming model for this problem.

5.4 ARTIFICIAL VARIABLE TECHNIQUES

5.4.1 Introduction:

The LPPs in which constraints may also have ≥ and = signs after ensuring that all $b_i \geq 0$ are considered in this section. In such cases basic matrix cannot be determined as an identify matrix in the starting simplex data therefore we introduce a new type of variable called the artificial variable. These variables are fictitious and cannot have any physical meaning. The artificial variable technique is merely a device to get the starting basic feasible solution, so that the simplex procedure may be adopted as usual until the optimal solution is obtained. To solve such LPP's there are two methods,

[i] The charle's Big – M Method or method of penalties.

[ii] The two phase simplex method.

5.4.2 The CHARNE'S BIG – M method:

The following steps are involved in solving an LPP using Big - M method.

Step 1: Express the problem in standard form.

Step 2 : Add non – negative artificial variables to the left side of each of the equations corresponding to constraints of the type ≥ or = . However, addition of these artificial variables causes. Violation of the corresponding constraints. Therefore, we would get rid of these variables and would not allow them to appear in the final solution. This is achieved by assigning a very large penalty [-M for maximization and M for Minimization in the objective function.

Step 3: Solve the modified LPP by simplex method, until any one of the three cases may arise.

[i] If no artificial variable appears in the basis and the quality optimality conditions are satisfied, then the current solution is an optimal basic feasible solution.

[ii] If atleast one artificial variable is there in the basis at zero level and optimality condition is satisfied, then the current solution is an optimal feasible basic solution.

[iii] If atleast one artificial variable appears in the basis at positive level and the optimality condition are satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize the objective function.

Note:

While applying simplex method, whenever an artificial variable happens to leave the basis drop that artificial variable and omit all the entries committed corresponding to its column from the simplex tab.

5.4.3 Two Phase Method

The Two Phase Method is another approach to handle situations where artificial variables are needed, and it avoids the use of a large penalty (Big M). Steps:

1. **Phase I**: Solve an auxiliary linear programming problem to minimize the sum of artificial variables. The objective is to drive all artificial variables to zero. If the minimum value is zero, the original problem is feasible; otherwise, it is infeasible.
2. **Phase II**: Once all artificial variables are zero, the original objective function is solved using the Simplex method.

This method is preferred when the Big M method is impractical due to the large values used for the Big M constants.

Example: 1: use penalty method to maximize $Z = 3x_1 + 2x_2$

Subject to the constraints,

$$2x_1 + x_2 \le 2$$
$$3x_1 + 4x_2 \ge 12$$
$$x_1, x_2 \ge 0$$

Solution:

By introducing slack variables $S_1 \geq 0$, surplus variable $S_2 \geq 0$ and artificial variable $A_1 \geq 0$,given LPP can be reformulated as,

Maximize $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - MA_1$

Subject to,

$$2x_1 + x_2 + s_1 = 2$$
$$3x_1 + 4x_2 - s_2 + A_1 = 12$$

The starting feasible solution is $S_1 = 2$, $A_1 = 12$

Initial table :

		C_1	3	2	0	0	-M	
C_B	B	X_B	X_1	X_2	S_1	S_2	A_1	X_B/X_2
0	S_1	2	2	1	1	0	0	2/1=2
-M	A_1	12	3	4	0	-1	1	12/4=3
	Z_j	-12M	-3M	-4M	0	M	-M	
	$Z_J – C_j$	-	-3M-3	-4M-2	0	M	0	

Since some of the $Z_J - C_j \leq 0$, the current feasible solution is not optimum . choose the most negative $Z_J - C_j$ = -4M – 2 .

X_2 variable enters the basis, and the basic value of variable S_1 leaves the basis.

FIRST ITERATION:

		C_1	3	2	0	0	-M
C_B	B	X_B	X_1	X_2	S_1	S_2	A_1
2	X_2	2	2	1	1	0	0
-M	A_1	4	-5	0	-4	-1	1
	Z_j	4-4M	4+5M	2	2+4M	M	-M
	$Z_J - C_j$	-	5M+1	0	4M+2	M	0

Since all $Z_J - C_j \geq 0$ and an artificial variable appears in the basis at positive level ,the given LPP does not possess any feasible solution . But the LPP possess a pseudo optimal solution.

Example : 2 : Solve the LPP

Minimize $Z = 4x_1 + x_2$

Subject to,

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

Solution:

Since the objective function is minimization, we covert it into maximization using

Minimize $Z = -Max(-Z) = -Max(Z^*)$

Maximize $Z^* = -4x_1 - x_2$

Subject to,

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Convert the given LPP into solution of standard by an adding artificial variables A_1 , A_2 , surplus variables, slack variables S_2 to get initial basic feasible solution.

Minimize $Z^* = -4x_1 - x_2 + 0s_1 + 0s_2 - MA_1 - MA$

Subject to, $3x_1 + x_2 + A_1 = 3$

The starting feasible value is $A_1 = 3, A_2 = 6, S_2 = 4$.

Initial solution:

		C_j	-4	-1	-M	0	-M
C_B	B	X_B	X_1	X_2	A_1	S_1	A_2
-M	A_1	2	3	1	1	0	0
-M	A_2	6	4	3	0	-1	1
⍰0	S_2	4	1	2	0	0	0
	Z_j	-9M	-7M	-4M	-M	-M	M
	$Z_J - C_j$	-	-7M+4	-4M+1	0	-M	2M

Since some of the $Z_J - C_j \leq 0$,the current feasible solution is not optimum. As X_1 enters the basis and the basic variables A_1 leaves the basis.

FIRST ITERATION :

		C_j	-4	-1	-M	0	-M	0	
C_B	B	X_B	X_1	X_2	A_1	S_1	A_2	S_2	Min XB/X_1
-M	A_1	3/2	5/2	0	1	0	0	-1/2	3/5
⍰-M	A_2	3/2	5/2	0	0	-1	1	-3/2	3/5
-1	A_3	3/2	½	1	0	0	0	½	3
	Z_j	-3M-3/2	-5M-1/2	-1	-M	+M	-M	2M-1/2	
	$Z_J - C_j$	-	-5M+1/2	0	0	M	0	2M-1/2	

Since $Z_J - C_j$ is negative ,the current feasible solution is not optimum. Therefore X_1 variable enters the basis and the artificial variables A_2 leaves the basis.

SECOND ITERATION :

		C_j	-4	-1	-M	0	0	
C_B	B	X_B	X_1	X_2	A_1	S_1	S_2	Min XB/X_2
⍰-M	A_1	0	0	0	1	1	1	0
-4	X_1	3/5	1	0	0	-2/5	-3/5	-
-1	X_2	6/5	0	1	0	-1/5	4/5	6/5
	Z_j	-18/5	-4	-1	-M	-M+9/5	-M+8/5	
	$Z_J - C_j$	-	0	0	0	-M+9/5	-M+8/5	

Since $Z_4 - C_4$ is most negative ,S_1 enters the basis and the artificial variables A_1 leaves the basis.

THIRD ITERATION :

C_j -4 -1 0 0

C_B	B	X_B	X_1	X_2	S_1	S_2	
0	S_1	0	0	0	1	1	0
-4	X_1	3/5	1	0	0	-1/5	-
-1	X_2	6/5	0	1	0	1	6/5
	Z_j	-18/5	-4	-1	0	-1/5	
	$Z_J - C_j$	-	0	0	0	-1/5	

Since $Z_4 - C_4$ is most negative ,S_2 enters the basis and S_1 leaves the basis.

FOURTH ITERATION :

C_j

C_B	B	X_B	X_1	X_2	S_1	S_2
0	S_2	0	0	0	1	1
-4	X_1	3/5	1	0	1/5	0
-1	X_2	6/5	0	1	-1	0
	Z_j	-18/5	-4	-1	1/5	0
	$Z_J - C_j$	-	0	0	1/5	0

Since all $Z_J - C_j \geq 0$ the solution is optimum and is given by ,$X_1 = 3/5$, $X_2 = 6/5$ and

Max Z=-18/5

Min Z =Max [-z] +18/5

Example : 3 : Solve by Big M – Method.

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$

Subject to,

$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$

Solution:

Since the constraints are equations , introduce artificial variables $A_1, A_2 \geq 0$.The reformulated problem is given as follows,

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$ Subject to ,

$$x_1 + 2x_2 + 3x_3 + A_1 = 15$$
$$2x_1 + x_2 + 5x_3 + A_2 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$

Initial solution is given by $A_1 = 15, A_2 = 20,\ and\ x_4 = 10,$

		C_j	1	2	3	-1	-M	-M	
C_B	B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	Min XB/X_3
-M	A_1	15	1	2	3	0	1	0	15/3=5
-M	A_2	20	2	1	5	0	0	1	20/5=4
-1	X_4	10	1	2	1	1	0	0	10/1=10
	Z_j	-35M	-3M	-3M	-8M	-1	-M	-M	
	$Z_J - C_j$	-10	-1 -3M-2	-2 -3M-4	-1 -8M-4	0	0	0	

Since $Z_3 - C_3$ is most negative , X_3 enters the basis and basic variable A_2 leaves the basis.

FIRST ITERATON :

		C_j	1	2	3	-1	-M	
C_B	B	X_B	X_1	X_2	X_3	X_4	A_1	Min XB/X_3
-M	A_1	3	-1/5	7/5	0	0	1	3/7/5=15/7
3	X_3	4	2/5	1/5	1	0	0	4/1/5=20/1
-1	X_4	6	3/5	9/5	0	1	0	6/9/5=30/9
	Z_j	- 35M+6	1/5M+3/5	-7/5M-6/5	3	-1	-M	
	$Z_J - C_j$		1/5M-2/5	-7/5M-16/5	0	0	0	

Since $Z_2 - C_2$ is most negative , X_2 enters the basis and the basic variable A_1 leaves the basis.

SECOND ITERATION :

		C_j	1	2	3	-1	
C_B	B	X_B	X_1	X_2	X_3	X_4	Min XB/X_1
2	X_2	15/7	-1/7	1	0	0	-
3	X_3	25/7	3/7	0	1	0	25/3
-1	X_4	15/7	6/7	0	0	1	15/6
	Z_j	90/7	1/7	2	3	-1	
	$Z_J - C_j$	-	-6/7	0	0	0	

Since $Z_1 - C_1 = -6/7$ is most negative , the current feasible solution is not optimum. Therefore X_1 enters the basis and the variables X_4 leaves the basis.

THIRD ITERATION :

		C_j	1	2	3	-1
C_B	B	X_B	X_1	X_2	X_3	X_4
2	X_2	15/6	0	1	0	1/6
3	X_3	15/6	0	0	1	3/6
3	X_1	15/6	1	0	0	7/6
	Z_j	15	1	2	3	3
	$Z_J - C_j$		0	0	0	4

Since $Z_j - C_j \geq 0$, the solution is optimum and is given by $X_1 = X_2 = X_3 = 15/6 = 5/2$ and max $z = 15$.

Example: 4 Use penalty method to solve the following LPP.

Minimize $Z = 5x + 3y$

Subject to,

$$2x + 4y \leq 12$$
$$2x + 2y = 10$$
$$5x + 2y \geq 10$$
$$x, y \geq 0$$

Solution:

First we convert the objective function from minimum to maximum using ,

Min $Z = -Max(-z)$

Rewrite the given LPP into standard form by adding slack variable $S_1 \geq 0$, surplus variable $S_2 \geq 0$ and the artificial variables $A_1, A_2 \geq 0$

$$Z = -5x - 3y + 0s_1 + 0s_2 - MA_1 - MA_2$$

Maximize

$$2x + 4y + s_1 = 12$$
$$2x + 2y + A_1 = 10$$
$$5x + 2y - s_2 + A_2 = 10$$
$$x, y, s_1, s_2, A_1, A_2 \geq 0$$

Initial feasible solution is given by $s_1 = 12, A_1 = 10, \mathit{and}\ A_2 = 10$, Since all $Z_j - C_j \geq 0$, and no artificial value in the basis, the solution is opinion and given by $X = 4$, $y = 1$, Max $z = -23$.

C_j -5 -3 0 -M 0 -M

C_B	B	X_B	X	y	S_1	A_1	S_2	A_2	Min XB/X
0 -M -M	S_1 A_1 A_2	12 10 10	2 2 5	4 2 2	1 0 0	0 1 0	0 0 -1	0 0 1	12/2=6 10/2=5 10/5=2
	Z_j $Z_J - C_j$	20M	-7M -7M+5	-4M -4M+3	0 0	-M 0	M M	-M 0	Min XB/y
0 -M -5	S_1 A_1 X	8 6 2	0 0 1	16/5 6 / 5 2/5	1 0 0	0 1 0	2/5 2/5 -1/5	 - -	8x5/10=3 ½ 30/=5 10/2=5
	Z_j $Z_J - C_j$	- 6M - 10	-5 0	-6/5M-2 - 6/5M+1	0 0	-M-2/5 0-2/5	M+1 M+1	 -	
-3 -M -5	Y A_1 X	5/2 3 1	0 0 1	1 0 0	5/16 -3/8 -1/8	0 1 0	1/8 -1/4 -1/4	-	
	Z_j $Z_J - C_j$	-3M- 25/2	-5 0	-3 0	-3/8M- 5/16 3/8M -5/16	-M 0 0	7/8 -1/4M 7/8 -M/4	- -	
-3 0 -5	Y S_2 X	1 12 4	0 0 1	1 0 0	½ -1.5 -0.5	- - -	0 1 0	- - -	
	Z_j $Z_J - C_j$	-23	-5 0	-3 0	1 1	- -	0 0	- -	

Exercises

1. Solve the following L.P.P using big-M method

   Minimize $Z = 10x_1 + 15x_2 + 20x_3$

   Subject to ,

   $2x_1 + 4x_2 + 6x_3 \geq 24$

   $3x_1 + 9x_2 + 6x_3 \geq 30$

   $x_1, x_2, x_3 \geq 0$

   Answer: $Z = 82, x_1 = 0, x_2 = \frac{6}{5}, x_3 = \frac{16}{5}$

2. Solve the following L.P.P using big-M method

Minimize $Z = 3x_1 + 2x_2 + 3x_3$

Subject to ,

$$2x_1 + x_2 + x_3 \ \square\ 2$$
$$3x_1 + 4x_2 + 2x_3 \ \square\ 8$$
$$x_1, x_2, x_3 \ \square\ 0$$

Answer: $Z = 4, x_1 = 0, x_2 = 2, x_3 = 0$

3. Solve the following L.P.P using two –phase method

Minimize $Z = x_1 + x_2$

Subject to ,

$$2x_1 + x_2 \ \square\ 4$$
$$x_1 + 7x_2 \ \square\ 7$$
$$x_1, x_2 \ \square\ 0$$

Answer: $Z = \frac{31}{13}, x_1 = \frac{21}{13}, x_2 = \frac{10}{13}$

4. Solve the following L.P.P using two –phase method

Minimize $Z = 5x_1 + 8x_2$

Subject to ,

$$3x_1 + 2x_2 \ \square\ 3$$
$$x_1 + 4x_2 \ \square\ 4$$
$$x_1 + x_2 \ \square\ 5$$
$$x_1, x_2 \ \square\ 0$$

Answer: $Z = 40, x_1 = 0, x_2 = 5$